Plurisubharmonic Envelopes and Supersolutions

Joint work with Chinh H. Lu and V. Guedj

Ahmed Zeriahi

Institut de Mathématiques de Toulouse (IMT)

Second Colloque des Mathmaticiens Marocains,
Université Cadi Ayad, Faculté des Sciences,
Marrakech, 26-29 Septembre, 2018

Introduction

Envelopes are classical objects in Convex Analysis, Potential Theory were Perron's method (1920) was used to solve the Dirichlet Problem for the Laplace (Poisson) equation. They also appear as solutions to (roof top) obstacle problems and free boundary value problems for second order (degenerate) elliptic PDE's.

Introduction

Envelopes are classical objects in Convex Analysis, Potential Theory were Perron's method (1920) was used to solve the Dirichlet Problem for the Laplace (Poisson) equation. They also appear as solutions to (roof top) obstacle problems and free boundary value problems for second order (degenerate) elliptic PDE's.

They were introduced in Complex Analysis by H. Bremermann (1959), and studied by J. Siciak (1962), V. Zaharyuta (1974) and used later in Pluripotential Theory by E. Bedford and B.A. Taylor in [BT76] to solve the Dirichlet problem for the complex Monge-Ampère equation.

Introduction

Envelopes are classical objects in Convex Analysis, Potential Theory were Perron's method (1920) was used to solve the Dirichlet Problem for the Laplace (Poisson) equation. They also appear as solutions to (roof top) obstacle problems and free boundary value problems for second order (degenerate) elliptic PDE's.

They were introduced in Complex Analysis by H. Bremermann (1959), and studied by J. Siciak (1962), V. Zaharyuta (1974) and used later in Pluripotential Theory by E. Bedford and B.A. Taylor in [BT76] to solve the Dirichlet problem for the complex Monge-Ampère equation.

Recently envelopes with obstacle were used successfully in Kähler Geometry by many authors in differents contexts: Boucksom-Berman (2008), Berman-Demailly (2009), Roos-Witt-Nystöm (2012), Boucksom-Bermann-Guedj-Zeriahi (2013), Berman (2013), Darvas (2014),Dravas-Rubinstein, etc...

Introduction

Envelopes are classical objects in Convex Analysis, Potential Theory were Perron's method (1920) was used to solve the Dirichlet Problem for the Laplace (Poisson) equation. They also appear as solutions to (roof top) obstacle problems and free boundary value problems for second order (degenerate) elliptic PDE's.

They were introduced in Complex Analysis by H. Bremermann (1959), and studied by J. Siciak (1962), V. Zaharyuta (1974) and used later in Pluripotential Theory by E. Bedford and B.A. Taylor in [BT76] to solve the Dirichlet problem for the complex Monge-Ampère equation.

Recently envelopes with obstacle were used successfully in Kähler Geometry by many authors in differents contexts: Boucksom-Berman (2008), Berman-Demailly (2009), Roos-Witt-Nystöm (2012), Boucksom-Bermann-Guedj-Zeriahi (2013), Berman (2013), Darvas (2014),Dravas-Rubinstein, etc...

Following the viscosity approach developed by P.-L. Lions and his collaborators in the early eighties ([CIL92]), we introduced in [EGZ11] viscosity sub/supper-solutions and used the Perron method to solve some degenerate complex Monge-Ampère equations on compact Kähler manifolds (without using Yau's theorem).

Following the viscosity approach developed by P.-L. Lions and his collaborators in the early eighties ([CIL92]), we introduced in [EGZ11] viscosity sub/supper-solutions and used the Perron method to solve some degenerate complex Monge-Ampère equations on compact Kähler manifolds (without using Yau's theorem).

The plan of this lecture is the following :

- explain how to extend Berman's approximation process of envelopes with obstacles by solutions to degenerate complex Monge-Ampère equations,

Following the viscosity approach developed by P.-L. Lions and his collaborators in the early eighties ([CIL92]), we introduced in [EGZ11] viscosity sub/supper-solutions and used the Perron method to solve some degenerate complex Monge-Ampère equations on compact Kähler manifolds (without using Yau's theorem).

The plan of this lecture is the following :

- explain how to extend Berman's approximation process of envelopes with obstacles by solutions to degenerate complex Monge-Ampère equations,
- explain the link between viscosity supersolutions and pluripotential supersolutions using envelopes,

Following the viscosity approach developed by P.-L. Lions and his collaborators in the early eighties ([CIL92]), we introduced in [EGZ11] viscosity sub/supper-solutions and used the Perron method to solve some degenerate complex Monge-Ampère equations on compact Kähler manifolds (without using Yau's theorem).

The plan of this lecture is the following :

- explain how to extend Berman's approximation process of envelopes with obstacles by solutions to degenerate complex Monge-Ampère equations,
- explain the link between viscosity supersolutions and pluripotential supersolutions using envelopes,
- solve degenerate complex Monge-Ampère equations using lower envelopes of pluripotential supersolutions and give a geometric application to the existence a singular Kähler-Einstein metric on certain algebraic varieties with mild singularities.

Quasi-plurisubharmonic functions

Let (X, ω) be a compact Kähler manifold of complex dimension $n \geq 1$. We will consider Kähler forms in the same (de Rham) cohomology class as ω.

Quasi-plurisubharmonic functions

Let (X, ω) be a compact Kähler manifold of complex dimension $n \geq 1$. We will consider Kähler forms in the same (de Rham) cohomology class as ω.

By the dd^c-lemma, any Kähler form $\tilde{\omega}$ in (the de Rham) cohomology class $\{\omega\}$ can be written as $\tilde{\omega} = \omega + dd^c\varphi$, where φ is a smooth real function on X such that $\tilde{\omega} = dd^c\varphi + \omega > 0$. This means that $\omega_\varphi := dd^c\varphi + \omega$ is a Kähler metric on X: the functions φ is called a Kähler potential for ω_φ.

Quasi-plurisubharmonic functions

Let (X, ω) be a compact Kähler manifold of complex dimension $n \geq 1$. We will consider Kähler forms in the same (de Rham) cohomology class as ω.

By the dd^c-lemma, any Kähler form $\tilde{\omega}$ in (the de Rham) cohomology class $\{\omega\}$ can be written as $\tilde{\omega} = \omega + dd^c\varphi$, where φ is a smooth real function on X such that $\tilde{\omega} = dd^c\varphi + \omega > 0$. This means that $\omega_\varphi := dd^c\varphi + \omega$ is a Kähler metric on X: the functions φ is called a Kähler potential for ω_φ.

A function $\varphi : X \longrightarrow \mathbb{R} \cup \{-\infty\}$ is said to be ω-plurisubharmonic if it is quasi-plurisbubharmonic in X and the associated curvature current $\omega_\varphi := \omega + dd^c\varphi \geq 0$ a closed positive currents on X.

Quasi-plurisubharmonic functions

Let (X, ω) be a compact Kähler manifold of complex dimension $n \geq 1$. We will consider Kähler forms in the same (de Rham) cohomology class as ω.

By the dd^c-lemma, any Kähler form $\tilde{\omega}$ in (the de Rham) cohomology class $\{\omega\}$ can be written as $\tilde{\omega} = \omega + dd^c\varphi$, where φ is a smooth real function on X such that $\tilde{\omega} = dd^c\varphi + \omega > 0$. This means that $\omega_\varphi := dd^c\varphi + \omega$ is a Kähler metric on X: the functions φ is called a Kähler potential for ω_φ.

A function $\varphi : X \longrightarrow \mathbb{R} \cup \{-\infty\}$ is said to be ω-plurisubharmonic if it is quasi-plurisbubharmonic in X and the associated curvature current $\omega_\varphi := \omega + dd^c\varphi \geq 0$ a closed positive currents on X.

Since we can write locally $\omega = dd^c\rho$, we have $\omega + dd^c\varphi = dd^c u$ where $u : \varphi + \rho$ is then (a local) plurisubharmonic function.
We denote by $PSH(X, \omega)$ the convex set of ω-plurisubharmonic functions in X. Then $PSH(X, \omega) \subset L^1(X)$.

Moreover by the dd^c-lemma we have a canonical isomorphism

$$PSH_0(X,\omega) := \{\varphi \in PSH(X,\omega); \max_X \varphi = 0\} \simeq \mathcal{T}_\omega^+(X),$$

where the right hand side denotes the convex set of closed positive $(1,1)$-currents on X that are cohomologuous to ω (fixed mass implies compactness of $PSH_0(X,\omega)$).

Moreover by the dd^c-lemma we have a canonical isomorphism

$$PSH_0(X,\omega) := \{\varphi \in PSH(X,\omega); \max_X \varphi = 0\} \simeq \mathcal{T}^+_\omega(X),$$

where the right hand side denotes the convex set of closed positive $(1,1)$-currents on X that are cohomologuous to ω (fixed mass implies compactness of $PSH_0(X,\omega)$).

Using Bedford and Taylor approach ([BT76,82]), it is possible to define the complex Monge-Ampère operator on $PSH(X,\omega) \cap L^\infty(X)$. Given $\varphi \in PSH(X,\omega) \cap L^\infty(X)$ we define locally in a coordinate chart $MA_\omega(\varphi) := (dd^c u)^n_{BT} = (\omega + dd^c\varphi)^n$, where $u := \varphi + \rho$ is a bounded plurisubharmonic function, ρ being a local potential of ω in the given chart.

Moreover by the dd^c-lemma we have a canonical isomorphism

$$PSH_0(X,\omega) := \{\varphi \in PSH(X,\omega); \max_X \varphi = 0\} \simeq \mathcal{T}^+_\omega(X),$$

where the right hand side denotes the convex set of closed positive $(1,1)$-currents on X that are cohomologuous to ω (fixed mass implies compactness of $PSH_0(X,\omega)$).

Using Bedford and Taylor approach ([BT76,82]), it is possible to define the complex Monge-Ampère operator on $PSH(X,\omega) \cap L^\infty(X)$. Given $\varphi \in PSH(X,\omega) \cap L^\infty(X)$ we define locally in a coordinate chart $MA_\omega(\varphi) := (dd^c u)^n_{BT} = (\omega + dd^c\varphi)^n$, where $u := \varphi + \rho$ is a bounded plurisubharmonic function, ρ being a local potential of ω in the given chart.

Then we can see $MA_\omega(\varphi)$ as positive Borel measure with total mass

$$\int_X MA_\omega(\varphi) = \int_X \omega^n_\varphi = \int_X \omega^n =: Vol(X,\omega).$$

Envelopes

Let $h : X \longrightarrow \mathbb{R}$ be a Borel function which is (upper) bounded, the obstacle function. We define the ω-psh envelope of h by the following formula:

$$P_\omega h := \left(\sup\{u(z); u \in PSH(X, \omega), u \leq h \text{ in } X\}\right)^* .$$

Envelopes

Let $h : X \longrightarrow \mathbb{R}$ be a Borel function which is (upper) bounded, the obstacle function. We define the ω-psh envelope of h by the following formula:

$$P_\omega h := \left(\sup\{u(z); u \in PSH(X,\omega), u \leq h \text{ in } X\}\right)^*.$$

By Bedford and Taylor ([1982]) we have $P_\omega h \leq h$ quasi everywhere (q.e.) in X and then

$$P_\omega h := \sup\{u(z); u \in PSH(X,\omega), u \leq h \text{ q.e.in } X\},$$

As we will see, this function is a solution to an obstacle problem in X.

Envelopes

Let $h : X \longrightarrow \mathbb{R}$ be a Borel function which is (upper) bounded, the obstacle function. We define the ω-psh envelope of h by the following formula:

$$P_\omega h := (\sup\{u(z); u \in PSH(X, \omega), u \leq h \text{ in } X\})^* .$$

By Bedford and Taylor ([1982]) we have $P_\omega h \leq h$ quasi everywhere (q.e.) in X and then

$$P_\omega h := \sup\{u(z); u \in PSH(X, \omega), u \leq h \text{ q.e.in } X\},$$

As we will see, this function is a solution to an obstacle problem in X.

To study the regularity of the upper envelope $P_\omega h$ in terms of the regularity of h, R. Berman ([Be13]) introduced a new and original approximation process that allows him to prove that $P_\omega h$ is almost $C^{1,1}(X)$ when h is smooth and ω is an entire cohomology class.

Here we are interested in more degenerate situations that arize in applications.

For our purpose we will need to define more general envelopes. Let μ be a non pluripolar positive Borel measure on X i.e. it puts no mass on pluripolar sets in X. The envelope of an (upper) bounded Borel function h on X with respect to μ is defined as

$$P_{\omega,\mu} h := \sup\{\phi; \phi \in PSH(X, \omega), \phi \leq h \ \ \mu - a.e. \text{ in } X\}.$$

Here we are interested in more degenerate situations that arize in applications.

For our purpose we will need to define more general envelopes. Let μ be a non pluripolar positive Borel measure on X i.e. it puts no mass on pluripolar sets in X. The envelope of an (upper) bounded Borel function h on X with respect to μ is defined as

$$P_{\omega,\mu} h := \sup\{\phi; \phi \in PSH(X,\omega), \phi \leq h \ \ \mu - a.e. \text{ in } X\}.$$

Here are some basic properties:

- $P_{\omega,\mu} h$ is ω-psh function in X,

Here we are interested in more degenerate situations that arize in applications.

For our purpose we will need to define more general envelopes. Let μ be a non pluripolar positive Borel measure on X i.e. it puts no mass on pluripolar sets in X. The envelope of an (upper) bounded Borel function h on X with respect to μ is defined as

$$P_{\omega,\mu}h := \sup\{\phi; \phi \in PSH(X,\omega), \phi \leq h \ \ \mu - a.e. \text{ in } X\}.$$

Here are some basic properties:

- $P_{\omega,\mu}h$ is ω-psh function in X,
- $P_{\omega,\mu}h \leq h$, μ-a.e. in X,

Here we are interested in more degenerate situations that arize in applications.

For our purpose we will need to define more general envelopes. Let μ be a non pluripolar positive Borel measure on X i.e. it puts no mass on pluripolar sets in X. The envelope of an (upper) bounded Borel function h on X with respect to μ is defined as

$$P_{\omega,\mu} h := \sup\{\phi ; \phi \in PSH(X,\omega), \phi \leq h \ \ \mu - a.e. \text{ in } X\}.$$

Here are some basic properties:

- $P_{\omega,\mu} h$ is ω-psh function in X,
- $P_{\omega,\mu} h \leq h$, μ-a.e. in X,
- $P_{\omega,\mu} h = P_\omega h$ in X, if $\mu = fdV$ is a volume form with a positive density $0 < f \in L^1(X)$,

Here we are interested in more degenerate situations that arize in applications.

For our purpose we will need to define more general envelopes. Let μ be a non pluripolar positive Borel measure on X i.e. it puts no mass on pluripolar sets in X. The envelope of an (upper) bounded Borel function h on X with respect to μ is defined as

$$P_{\omega,\mu} h := \sup\{\phi; \phi \in PSH(X,\omega), \phi \leq h \ \ \mu - a.e. \text{ in } X\}.$$

Here are some basic properties:

- $P_{\omega,\mu} h$ is ω-psh function in X,
- $P_{\omega,\mu} h \leq h$, μ-a.e. in X,
- $P_{\omega,\mu} h = P_\omega h$ in X, if $\mu = fdV$ is a volume form with a positive density $0 < f \in L^1(X)$,
- $P_{\omega,\mu}\psi = \psi$ when $\psi \in PSH(X,\omega)$, hence $P_{\omega,\mu} \circ P_{\omega,\mu} = P_{\omega,\mu}$,
- $\|P_{\omega,\mu} h_1 - P_{\omega,\mu} h_2\|_{L^\infty(X,\mu)} \leq \|h_1 - h_2\|_{L^\infty(X,\mu)}$,

Here we are interested in more degenerate situations that arize in applications.

For our purpose we will need to define more general envelopes. Let μ be a non pluripolar positive Borel measure on X i.e. it puts no mass on pluripolar sets in X. The envelope of an (upper) bounded Borel function h on X with respect to μ is defined as

$$P_{\omega,\mu}h := \sup\{\phi; \phi \in PSH(X,\omega), \phi \leq h \;\; \mu - a.e. \text{ in } X\}.$$

Here are some basic properties:

- $P_{\omega,\mu}h$ is ω-psh function in X,
- $P_{\omega,\mu}h \leq h$, μ-a.e. in X,
- $P_{\omega,\mu}h = P_\omega h$ in X, if $\mu = fdV$ is a volume form with a positive density $0 < f \in L^1(X)$,
- $P_{\omega,\mu}\psi = \psi$ when $\psi \in PSH(X,\omega)$, hence $P_{\omega,\mu} \circ P_{\omega,\mu} = P_{\omega,\mu}$,
- $\|P_{\omega,\mu}h_1 - P_{\omega,\mu}h_2\|_{L^\infty(X,\mu)} \leq \|h_1 - h_2\|_{L^\infty(X,\mu)}$,
- If (h_j) decreases to h, then $P_{\omega,\mu}(h_j)$ decreases to $P_{\omega,\mu}h$.

Here we are interested in more degenerate situations that arize in applications.

For our purpose we will need to define more general envelopes. Let μ be a non pluripolar positive Borel measure on X i.e. it puts no mass on pluripolar sets in X. The envelope of an (upper) bounded Borel function h on X with respect to μ is defined as

$$P_{\omega,\mu}h := \sup\{\phi; \phi \in PSH(X,\omega), \phi \leq h \ \ \mu - a.e. \text{ in } X\}.$$

Here are some basic properties:

- $P_{\omega,\mu}h$ is ω-psh function in X,
- $P_{\omega,\mu}h \leq h$, μ-a.e. in X,
- $P_{\omega,\mu}h = P_\omega h$ in X, if $\mu = fdV$ is a volume form with a positive density $0 < f \in L^1(X)$,
- $P_{\omega,\mu}\psi = \psi$ when $\psi \in PSH(X,\omega)$, hence $P_{\omega,\mu} \circ P_{\omega,\mu} = P_{\omega,\mu}$,
- $\|P_{\omega,\mu}h_1 - P_{\omega,\mu}h_2\|_{L^\infty(X,\mu)} \leq \|h_1 - h_2\|_{L^\infty(X,\mu)}$,
- If (h_j) decreases to h, then $P_{\omega,\mu}(h_j)$ decreases to $P_{\omega,\mu}h$.

The main property of the envelope is the following orthogonality property which plays an important role.

Proposition

If h is (quasi) lower semi-continuous in X and $\hat{h} := P_{\omega,\mu} h$, then

$$\int_X (h - \hat{h}) MA(\hat{h}) = 0.$$

In particular $\int_{\{\hat{h}<h\}} MA_\omega(\hat{h}) = 0$ and the Monge-Ampère measure of $\hat{h}$ is supported in the contact set $\{\hat{h} = h\}$.

This shows that $\hat{h}$ is a solution of an obstacle problem.The proof is based on a local balayage argument (see [BT82]).

Degenerate Complex Monge-Ampère Equations

Motivated by the problem of the existence of Kähler-Einstein metrics on compact Kähler varieties with mild singularities, we will consider a more general geometric context.

Degenerate Complex Monge-Ampère Equations

Motivated by the problem of the existence of Kähler-Einstein metrics on compact Kähler varieties with mild singularities, we will consider a more general geometric context.

Let $\theta \geq 0$ be a smooth closed semi-positive $(1,1)$-form on X such that $\int_X \theta^n > 0$ and let μ be a degenerate volume form on X which puts no mass on pluripolar subsets of X and such that $\int_X \mu = \int_X \theta^n$ e.g. $\theta := f^*(\omega_Y)$ and $\mu := f^*(dV_Y)$ where $f : X \longrightarrow Y$ is a resolution of a singular Kähler projective variety with mild singularities, $, \omega_Y$ is a Kḧaler metric on Y and dV_Y is a smooth non degenerate volume form on Y.

We need to extend the complex Monge-Ampère operator to a class of singular θ-psh functions.

Degenerate Complex Monge-Ampère Equations

Motivated by the problem of the existence of Kähler-Einstein metrics on compact Kähler varieties with mild singularities, we will consider a more general geometric context.
Let $\theta \geq 0$ be a smooth closed semi-positive $(1,1)$-form on X such that $\int_X \theta^n > 0$ and let μ be a degenerate volume form on X which puts no mass on pluripolar subsets of X and such that $\int_X \mu = \int_X \theta^n$ e.g. $\theta := f^*(\omega_Y)$ and $\mu := f^*(dV_Y)$ where $f : X \longrightarrow Y$ is a resolution of a singular Kähler projective variety with mild singularities, $,\omega_Y$ is a Kḧaler metric on Y and dV_Y is a smooth non degenerate volume form on Y.

We need to extend the complex Monge-Ampère operator to a class of singular θ-psh functions. More precisely for a given θ-psh function $\varphi \in PSH(X,\theta)$, we associate the non pluripolar part of the Monge-Ampère measure of φ, denoted by $\langle(\theta + dd^c\varphi)^n\rangle$, following an idea of Bedford and Taylor ([BT85]) in the local case.

This is defined by

$$\langle(\theta + dd^c\varphi)^n\rangle := \lim_{k\to+\infty} \mathbf{1}_{\{\varphi>-k\}}(\theta + dd^c \sup\{\varphi, -k\})^n.$$

By definition this is a positive Borel measure on X which is concentrated on the Borel set $\{\varphi > -\infty\}$ and puts no mass on pluripolar sets of X, whose total mass is $\leq \int_X \theta^n$.
Then we define $\mathcal{E}(X,\theta)$ to be the class of functions $\varphi \in PSH(X,\theta)$ with full Monge-Ampère mass i.e. such that $\int_X \langle(\theta + dd^c\varphi)^n\rangle = \int_X \theta^n$.

This is defined by

$$\langle(\theta + dd^c\varphi)^n\rangle := \lim_{k\to+\infty} \mathbf{1}_{\{\varphi>-k\}}(\theta + dd^c \sup\{\varphi, -k\})^n.$$

By definition this is a positive Borel measure on X which is concentrated on the Borel set $\{\varphi > -\infty\}$ and puts no mass on pluripolar sets of X, whose total mass is $\leq \int_X \theta^n$.
Then we define $\mathcal{E}(X, \theta)$ to be the class of functions $\varphi \in PSH(X, \theta)$ with full Monge-Ampère mass i.e. such that $\int_X \langle(\theta + dd^c\varphi)^n\rangle = \int_X \theta^n$.

This the largest class containing $PSH(X, \theta) \cap L^\infty(X)$ where the complex Monge-Ampère operator is well defined and satisfies the comparison principle.

Let us denote by $MA_\theta(\varphi) := \langle(\theta + dd^c\varphi)^n\rangle = (\theta + dd^c\varphi)^n$ the Monge-Ampère measure of φ.

We will need the following result.

Theorem (GZ07, EGZ09)

Let $\mu \geq 0$ be a non pluripolar positive Borel measure on X with $\mu(X) = \int_X \theta^n$. For any $\alpha \geq 0$, there exists a unique function $\varphi = \varphi_\alpha \in \mathcal{E}(X, \theta)$ (normalized when $\alpha = 0$) such that

$$MA_\theta(\varphi) = e^{\alpha\varphi}\mu, \tag{1}$$

weakly in the sense of currents on X.

When $\theta > 0$ is Kähler and $\mu > 0$ is a smooth non degenerate volume form on X, the theorem is well known.

- If $\alpha = 0$ by a famous theorem of S.T. Yau ([Y78]) the equation (1) has a unique (normalized) smooth solution.

We will need the following result.

Theorem (GZ07, EGZ09)

Let $\mu \geq 0$ be a non pluripolar positive Borel measure on X with $\mu(X) = \int_X \theta^n$. For any $\alpha \geq 0$, there exists a unique function $\varphi = \varphi_\alpha \in \mathcal{E}(X,\theta)$ (normalized when $\alpha = 0$) such that

$$MA_\theta(\varphi) = e^{\alpha\varphi}\mu, \tag{1}$$

weakly in the sense of currents on X.

When $\theta > 0$ is Kähler and $\mu > 0$ is a smooth non degenerate volume form on X, the theorem is well known.

- If $\alpha = 0$ by a famous theorem of S.T. Yau ([Y78]) the equation (1) has a unique (normalized) smooth solution.
- If $\alpha > 0$, it was proved independently by T. Aubin [A78] and by S.T. Yau ([Y78] that the equation (1) has a unique smooth solution.

These theorems answered some of the fundamental problems in Kähler Geometry posed by E. Calabi in the early fifthies.

In the degenerate case, when μ is a continuous positive volume form, it was proved in ([EGZ11]) that the solution to the complex Monge-Ampère equation (1) with $\alpha > 0$ can be also expressed as the upper envelope of all (pluripotential = viscosity) subsolutions of the equation (1) .

This follows from the comparison principle for the equation (1).

Proposition

Let $\varphi \in \mathcal{E}(X, \theta)$ be a pluripotential subsolution to the equation (1) with $\alpha > 0$, and let $\psi \in \mathcal{E}(X, \theta)$ be a pluripotential supersolution of the same equation. Then $\varphi \leq \psi$ in X.

We will now give a generalization of the Berman process of approximation of psh envelopes by solutions to complex Monge-Ampère equations.

Theorem (GLZ17)

Let u be a bounded lower semi-continuous function in X and let (φ_j) be the sequence of functions in $\mathcal{E}(X,\theta)$ solving the following equations

$$(\theta + dd^c \varphi_j)^n = e^{j(\varphi_j - u)}\mu,\ j \in \mathbb{N}. \tag{2}$$

Then the sequence converges in $L^1(X)$ and in capacity to the (θ,μ)-envelope of u defined by

$$P_{\theta,\mu} u := \sup\{\varphi \in PSH(X,\theta); \varphi \leq u, \mu - a.e. \text{ in } X\}.$$

Proof : We first give a lower bound of φ_j. Let $\psi \in \mathcal{E}(X, \theta)$ be the unique solution of the following equation $MA_\theta(\psi) = e^\psi \mu$. Set $\hat{u} := P_{\theta,\mu} u$. We claim that for any $j \geq 1$,

$$\varphi_j \geq (1 - 1/j)\hat{u} + (1/j)(\psi - n \log j + \inf_X u).$$

Proof : We first give a lower bound of φ_j. Let $\psi \in \mathcal{E}(X, \theta)$ be the unique solution of the following equation $MA_\theta(\psi) = e^\psi \mu$. Set $\hat{u} := P_{\theta,\mu} u$. We claim that for any $j \geq 1$,

$$\varphi_j \geq (1 - 1/j)\hat{u} + (1/j)(\psi - n \log j + \inf_X u).$$

Indeed it's enough to show that the right hand side is a pluripotential subsolution of the equation (2) and then apply the Comparison Principle.

Proof : We first give a lower bound of φ_j. Let $\psi \in \mathcal{E}(X, \theta)$ be the unique solution of the following equation $MA_\theta(\psi) = e^\psi \mu$. Set $\hat{u} := P_{\theta,\mu} u$. We claim that for any $j \geq 1$,

$$\varphi_j \geq (1 - 1/j)\hat{u} + (1/j)(\psi - n \log j + \inf_X u).$$

Indeed it's enough to show that the right hand side is a pluripotential subsolution of the equation (2) and then apply the Comparison Principle.

To conclude the proof we only need to show that $\varphi_j \to \hat{u}$ in $L^1(X)$. Thanks to the previous lower bound, it's enough to prove that $\limsup_j \varphi_j \leq \hat{u}$ in X.

Proof : We first give a lower bound of φ_j. Let $\psi \in \mathcal{E}(X, \theta)$ be the unique solution of the following equation $MA_\theta(\psi) = e^\psi \mu$. Set $\hat{u} := P_{\theta,\mu} u$. We claim that for any $j \geq 1$,

$$\varphi_j \geq (1 - 1/j)\hat{u} + (1/j)(\psi - n \log j + \inf_X u).$$

Indeed it's enough to show that the right hand side is a pluripotential subsolution of the equation (2) and then apply the Comparison Principle.

To conclude the proof we only need to show that $\varphi_j \to \hat{u}$ in $L^1(X)$. Thanks to the previous lower bound, it's enough to prove that $\limsup_j \varphi_j \leq \hat{u}$ in X.
Indeed let us fix $\varepsilon > 0$ and consider the sets

$$A_j := \{x \in X; \varphi_j(x) - u(x) \geq \varepsilon\}.$$

From the equation (2) we see that for any $j \geq 1$,

$$\mu(A_j) \leq e^{-\varepsilon j} \int_X e^{j(\varphi_j - u)} \mu \leq e^{-\varepsilon j} \int_X MA(\varphi_j) = e^{-\varepsilon j}.$$

From the equation (2) we see that for any $j \geq 1$,

$$\mu(A_j) \leq e^{-\varepsilon j} \int_X e^{j(\varphi_j - u)} \mu \leq e^{-\varepsilon j} \int_X MA(\varphi_j) = e^{-\varepsilon j}.$$

Then

$$\mu(\cup_{k \geq j} A_k) \leq \frac{e^{-\varepsilon j}}{1 - e^{-\varepsilon}}.$$

From the equation (2) we see that for any $j \geq 1$,

$$\mu(A_j) \leq e^{-\varepsilon j} \int_X e^{j(\varphi_j - u)} \mu \leq e^{-\varepsilon j} \int_X MA(\varphi_j) = e^{-\varepsilon j}.$$

Then

$$\mu(\cup_{k \geq j} A_k) \leq \frac{e^{-\varepsilon j}}{1 - e^{-\varepsilon}}.$$

This means that the set $A := \cap_{j \geq 1} (\cup_{k \geq j} A_k)$ satisfies $\mu(A) = 0$ and for any $x \in X \setminus A$, we have

$$\limsup_{j \to +\infty} \varphi_j(x) \leq \varepsilon.$$

From the equation (2) we see that for any $j \geq 1$,

$$\mu(A_j) \leq e^{-\varepsilon j} \int_X e^{j(\varphi_j - u)} \mu \leq e^{-\varepsilon j} \int_X MA(\varphi_j) = e^{-\varepsilon j}.$$

Then

$$\mu(\cup_{k \geq j} A_k) \leq \frac{e^{-\varepsilon j}}{1 - e^{-\varepsilon}}.$$

This means that the set $A := \cap_{j \geq 1}(\cup_{k \geq j} A_k)$ satisfies $\mu(A) = 0$ and for any $x \in X \setminus A$, we have

$$\limsup_{j \to +\infty} \varphi_j(x) \leq \varepsilon.$$

This sows that $\limsup_{j \to +\infty} \varphi_j(x) \leq u(x)$, μ-a.e. in X. Hence $(\limsup \varphi_j)^* \leq P_{\theta,\mu}$ in X.

Viscosity concepts

We fix a continuous positive function $f > 0$ in X and consider the following complex Monge-Ampère equation

$$(\theta + dd^c\varphi)^n = e^{\varphi} f dV. \tag{3}$$

Definition

1. A bounded upper semi-continuous function $u : X \longrightarrow \mathbb{R}$ is said to be a viscosity subsolution to the equation (3) if for any point $a \in X$ and any smooth local upper test function $q \geq_a u$ for u at the point a, we have

$$(\theta + dd^c q)^n(a) \geq e^{q(a)} f(a) dV(a).$$

Viscosity concepts

We fix a continuous positive function $f > 0$ in X and consider the following complex Monge-Ampère equation

$$(\theta + dd^c \varphi)^n = e^{\varphi} f dV. \tag{3}$$

Definition

1. A bounded upper semi-continuous function $u : X \longrightarrow \mathbb{R}$ *is said to be a viscosity subsolution to the equation (3) if for any point* $a \in X$ *and any smooth local upper test function* $q \geq_a u$ *for* u *at the point* a*, we have*

$$(\theta + dd^c q)^n(a) \geq e^{q(a)} f(a) dV(a).$$

2. A bounded lower semi-continuous function $v : X_T \to \mathbb{R}$ *is a viscosity supersolution to the equation (3) if for any point* $a \in X$ *and any smooth local lower test function* $q \leq_a v$ *for* v *at the point* a*, we have*

$$(\theta(a) + dd^c q(a))_+^n \leq e^{q(a)} f(a) dV(a).$$

The supersolution theorem

It's known that a bounded upper semi-continuous function $u : X \longrightarrow \mathbb{R}$ is a viscosity subsolution of the equation(3) iff u is a (θ-psh in X) subsolution of the equation (see [EGZ11]).

The supersolution theorem

It's known that a bounded upper semi-continuous function $u : X \longrightarrow \mathbb{R}$ is a viscosity subsolution of the equation(3) iff u is a (θ-psh in X) subsolution of the equation (see [EGZ11]).

Our second main result concerns supersolutions.

Theorem (GLZ17)

Let $\mu = fdV$ be a volume form with continuous positive density $f > 0$ and let $v : X \longrightarrow \mathbb{R}$ be a bounded lower semi-continuous function in X.
If v is a viscosity supersolution of the Monge-Ampère equation (3), then $\varphi := P_\theta v$ is a pluripotential supersolution to this equation.

The supersolution theorem

It's known that a bounded upper semi-continuous function $u : X \longrightarrow \mathbb{R}$ is a viscosity subsolution of the equation(3) iff u is a (θ-psh in X) subsolution of the equation (see [EGZ11]).

Our second main result concerns supersolutions.

Theorem (GLZ17)

Let $\mu = fdV$ be a volume form with continuous positive density $f > 0$ and let $v : X \longrightarrow \mathbb{R}$ be a bounded lower semi-continuous function in X. If v is a viscosity supersolution of the Monge-Ampère equation (3), then $\varphi := P_\theta v$ is a pluripotential supersolution to this equation.

Sketch of the proof : Apply the approximation theorem and get a sequence (φ_j) of solutions to the following equations

$$(\theta + dd^c \varphi_j)^n = e^{j(\varphi_j - v)} e^{\varphi_j} dV = e^{(j+1)(\varphi_j - v)} e^v f dV.$$

Then we know that $\varphi_j \to P_{\theta,\mu} v = P_\theta v$, where $\mu := e^v f dV$.
Observe that φ_j is a viscosity subsolution to the equation (3). By the viscosity Comparison Principle ([EGZ11], [EGZ17]) we conclude that $\varphi_j \leq v$ on a Zariski open subset of X, hence $\varphi_j \leq P_\theta v$ in X and then $(\theta + dd^c \varphi_j)^n \leq e^v f dV$ in the pluripotential sense on X.

Then we know that $\varphi_j \to P_{\theta,\mu} v = P_\theta v$, where $\mu := e^v f dV$.
Observe that φ_j is a viscosity subsolution to the equation (3). By the viscosity Comparison Principle ([EGZ11], [EGZ17]) we conclude that $\varphi_j \leq v$ on a Zariski open subset of X, hence $\varphi_j \leq P_\theta v$ in X and then $(\theta + dd^c \varphi_j)^n \leq e^v f dV$ in the pluripotential sense on X.

We know by the approximation theorem that actually (φ_j) converges in capacity to $P_\theta v$ and we have a uniform minorant in $\mathcal{E}(X, \theta)$.

Therefore we can pass to the limit and get $(\theta + dd^c \varphi_j)^n \to (\theta + dd^c P_\theta v)^n$ in the weak sense of currents on X, which implies that $(\theta + dd^c P_\theta v)^n \leq e^v f dV$ in the pluripotential sense on X.

Then we know that $\varphi_j \to P_{\theta,\mu} v = P_\theta v$, where $\mu := e^v f dV$.
Observe that φ_j is a viscosity subsolution to the equation (3). By the viscosity Comparison Principle ([EGZ11], [EGZ17]) we conclude that $\varphi_j \leq v$ on a Zariski open subset of X, hence $\varphi_j \leq P_\theta v$ in X and then $(\theta + dd^c \varphi_j)^n \leq e^v f dV$ in the pluripotential sense on X.

We know by the approximation theorem that actually (φ_j) converges in capacity to $P_\theta v$ and we have a uniform minorant in $\mathcal{E}(X, \theta)$.

Therefore we can pass to the limit and get $(\theta + dd^c \varphi_j)^n \to (\theta + dd^c P_\theta v)^n$ in the weak sense of currents on X, which implies that $(\theta + dd^c P_\theta v)^n \leq e^v f dV$ in the pluripotential sense on X.

We can actually prove that (φ_j) is a non decreasing sequence, and the conclusion follows from Bedford-Taylor convergence theorem.

The Minimum Principle

Here we want to consider the following degenerate complex Monge-Ampère equation

$$(\theta + dd^c\varphi)^n = e^{\varphi}\mu, \tag{4}$$

where μ is a positive non pluripolar Borel measure (volume form) on an open set $\Omega \subset X$ with $\mu(\Omega) = +\infty$. We may assume that $\Omega \subset \mathrm{Amp}\,(\theta)$.

The Minimum Principle

Here we want to consider the following degenerate complex Monge-Ampère equation

$$(\theta + dd^c\varphi)^n = e^{\varphi}\mu, \tag{4}$$

where μ is a positive non pluripolar Borel measure (volume form) on an open set $\Omega \subset X$ with $\mu(\Omega) = +\infty$. We may assume that $\Omega \subset \mathrm{Amp}\,(\theta)$.

We would like to solve the above equation by considering its "minimal supersolution".

Observe that if $u, v \in \mathcal{E}(X, \theta)$ are two pluripotential supersolutions of the equation (4) then $\inf(u, v)$ may not be a supersolution since it may not be θ-psh anymore. However we can prove the following fact :

The Minimum Principle

Here we want to consider the following degenerate complex Monge-Ampère equation

$$(\theta + dd^c \varphi)^n = e^{\varphi} \mu, \qquad (4)$$

where μ is a positive non pluripolar Borel measure (volume form) on an open set $\Omega \subset X$ with $\mu(\Omega) = +\infty$. We may assume that $\Omega \subset \mathrm{Amp}\,(\theta)$.

We would like to solve the above equation by considering its "minimal supersolution".

Observe that if $u, v \in \mathcal{E}(X, \theta)$ are two pluripotential supersolutions of the equation (4) then $\inf(u, v)$ may not be a supersolution since it may not be θ-psh anymore. However we can prove the following fact :

The Minimum Principle : If $u, v \in \mathcal{E}(X, \theta)$ are two pluripotential supersolutions of the equation (4), then the function $w := P_\theta(u, v) = P_\theta(\inf(u, v))$ is a pluripotential subsolution of the equation (4).

When u, v are smooth, this was proved by T. Darvas. Our proof uses the

Solution to degenerate CMA equations

Now we can state our third main result.

Theorem (GLZ17)

Assume that the equation (4) admits a subsolution i.e. there exists $u_0 \in \mathcal{E}(X, \theta)$ such that

$$(\theta + dd^c u_0)^n \geq e^{u_0} \mu,$$

in the weak sense of currents on X. Set

$$\varphi := \inf\{\psi; \psi \in \mathcal{E}(X, \theta), MA_\theta(\psi) \leq \mathbf{1}_\Omega e^\psi \mu\}.$$

Then $\varphi \in \mathcal{E}(X, \theta)$ is the unique solution to the complex Monge-Ampère equation (4) i.e. $MA_\theta(\varphi) = \mathbf{1}_\Omega e^\varphi \mu$.

Proof : 1) We first prove that there is at least one supersolution to the equation (4). Indeed let $K \subset X$ be a compact subset such that $0 < \mu(K) < +\infty$ and let $\phi_K \in \mathcal{E}(X, \theta)$ be a solution of the equation $MA_\theta(\phi_K) = e^{\phi_K} \mu_K$, where $\mu_K := \mathbf{1}_K \mu$ is finite non pluripolar positive Borel measure on X. Then ϕ_K is a pluripotential supersolution to the equation (4).

Proof : 1) We first prove that there is at least one supersolution to the equation (4). Indeed let $K \subset X$ be a compact subset such that $0 < \mu(K) < +\infty$ and let $\phi_K \in \mathcal{E}(X, \theta)$ be a solution of the equation $MA_\theta(\phi_K) = e^{\phi_K} \mu_K$, where $\mu_K := \mathbf{1}_K \mu$ is finite non pluripolar positive Borel measure on X. Then ϕ_K is a pluripotential supersolution to the equation (4).

2) Apply the comparison principle (which still holds in this general situation) to get the inequality $u_0 \leq \varphi \leq \varphi_K$ in X.

Proof : 1) We first prove that there is at least one supersolution to the equation (4). Indeed let $K \subset X$ be a compact subset such that $0 < \mu(K) < +\infty$ and let $\phi_K \in \mathcal{E}(X,\theta)$ be a solution of the equation $MA_\theta(\phi_K) = e^{\phi_K}\mu_K$, where $\mu_K := \mathbf{1}_K\mu$ is finite non pluripolar positive Borel measure on X. Then ϕ_K is a pluripotential supersolution to the equation (4).

2) Apply the comparison principle (which still holds in this general situation) to get the inequality $u_0 \leq \varphi \leq \varphi_K$ in X.
By classical arguments, φ is an infimum of a countable sequence of supersolutions $(\psi)_j$. By the minimum principle we can assume that the sequence $(\psi)_j$ is non decreasing and converges to $\varphi \geq u_0$ in X, hence $\varphi \in \mathcal{E}(X,\theta)$.

Proof : 1) We first prove that there is at least one supersolution to the equation (4). Indeed let $K \subset X$ be a compact subset such that $0 < \mu(K) < +\infty$ and let $\phi_K \in \mathcal{E}(X, \theta)$ be a solution of the equation $MA_\theta(\phi_K) = e^{\phi_K} \mu_K$, where $\mu_K := \mathbf{1}_K \mu$ is finite non pluripolar positive Borel measure on X. Then ϕ_K is a pluripotential supersolution to the equation (4).

2) Apply the comparison principle (which still holds in this general situation) to get the inequality $u_0 \leq \varphi \leq \varphi_K$ in X.
By classical arguments, φ is an infimum of a countable sequence of supersolutions $(\psi)_j$. By the minimum principle we can assume that the sequence $(\psi)_j$ is non decreasing and converges to $\varphi \geq u_0$ in X, hence $\varphi \in \mathcal{E}(X, \theta)$.
3) We need to show that $MA_\theta(\varphi) \leq \mathbf{1}_\Omega e^\varphi \mu$ in the pluripotential sense on X.

This is clear by the convergence theorem on the open set Ω since μ has locally finite mass in Ω. The delicate point is to show that $MA_\theta(\varphi)$ is concentrated in Ω. Here we will use the subsolution u_0 in a crucial way.

This is clear by the convergence theorem on the open set Ω since μ has locally finite mass in Ω. The delicate point is to show that $MA_\theta(\varphi)$ is concentrated in Ω. Here we will use the subsolution u_0 in a crucial way. Since $e^{u_0}\mu \leq MA_\theta(u_0)$ we dedude that the measure $\mu_0 := e^{u_0}\mu$ has finite mass in X and we have for any j

$$e^{u_0} MA_\theta(\psi_j) \leq \mathbf{1}_\Omega e^{\psi_j} \mu_0,$$

in the sens of Radon measures on X.

This is clear by the convergence theorem on the open set Ω since μ has locally finite mass in Ω. The delicate point is to show that $MA_\theta(\varphi)$ is concentrated in Ω. Here we will use the subsolution u_0 in a crucial way. Since $e^{u_0}\mu \leq MA_\theta(u_0)$ we dedude that the measure $\mu_0 := e^{u_0}\mu$ has finite mass in X and we have for any j

$$e^{u_0} MA_\theta(\psi_j) \leq \mathbf{1}_\Omega e^{\psi_j}\mu_0,$$

in the sens of Radon measures on X.

Now we can pass to the limit and get the inequality $e^{u_0} MA_\theta(\varphi) \leq \mathbf{1}_\Omega e^{\varphi}\mu_0$ in the sens of Radon measures on X, which imples that $MA_\theta(\varphi)$ is concentrated in Ω since $\mu(\{u_0 = -\infty\}) = 0$.

4) The last step is to show that $MA_\theta(\varphi) = \mathbf{1}_\Omega e^{\varphi}\mu$. This is a local property. Fix a small open set $B \Subset \Omega$. We will prove that $MA_\theta(\varphi) = e^{\varphi}\mu$ weakly on B. We construc a supersolution ψ smaller that φ which solves the equation on B. Then $\psi = \varphi$ will solve the equation on B.

This is usually done by a balayage process solving a local Dirichlet problem with boundary values φ on a small ball B and gluying the local solution with φ (delicate analysis near the boundary).

We will use a new global balayage method which is simpler. Namely we solve the equation

$$MA(\psi) = \mathbf{1}_{\Omega \setminus B} e^{\psi - \varphi)} MA(\varphi) + \mathbf{1}_B e^{\psi} \mu = e^{\psi} \nu, \tag{5}$$

This is usually done by a balayage process solving a local Dirichlet problem with boundary values φ on a small ball B and gluying the local solution with φ (delicate analysis near the boundary).

We will use a new global balayage method which is simpler. Namely we solve the equation

$$MA(\psi) = \mathbf{1}_{\Omega \setminus B} e^{\psi - \varphi)} MA(\varphi) + \mathbf{1}_B e^{\psi} \mu = e^{\psi} \nu, \tag{5}$$

where $\nu := \mathbf{1}_{\Omega \setminus B} e^{-\varphi} MA(\varphi) + \mathbf{1}_B \mu$ is a Borel measure which a priori need not have a finite mass.

This is usually done by a balayage process solving a local Dirichlet problem with boundary values φ on a small ball B and gluying the local solution with φ (delicate analysis near the boundary).

We will use a new global balayage method which is simpler. Namely we solve the equation

$$MA(\psi) = \mathbf{1}_{\Omega\setminus B} e^{\psi-\varphi)} MA(\varphi) + \mathbf{1}_B e^{\psi}\mu = e^{\psi}\nu, \tag{5}$$

where $\nu := \mathbf{1}_{\Omega\setminus B} e^{-\varphi} MA(\varphi) + \mathbf{1}_B \mu$ is a Borel measure which a priori need not have a finite mass.

However we can approximate ν by an increasing sequence of non pluripolar measures of finite masses on X defined by

$$\nu_j := \mathbf{1}_{\Omega\setminus B} e^{-\max(\varphi,-j)} MA(\varphi) + \mathbf{1}_B \mu$$

and solve the corresponding equations :

$$MA(\psi_j) = \mathbf{1}_{\Omega\setminus B} e^{\psi_j-\max(\varphi,-j))} MA(\varphi) + \mathbf{1}_B e^{\psi_j}\mu = e^{\psi_j}\nu. \tag{6}$$

This is usually done by a balayage process solving a local Dirichlet problem with boundary values φ on a small ball B and gluying the local solution with φ (delicate analysis near the boundary).

We will use a new global balayage method which is simpler. Namely we solve the equation

$$MA(\psi) = \mathbf{1}_{\Omega \setminus B} e^{\psi - \varphi)} MA(\varphi) + \mathbf{1}_B e^{\psi} \mu = e^{\psi} \nu, \tag{5}$$

where $\nu := \mathbf{1}_{\Omega \setminus B} e^{-\varphi} MA(\varphi) + \mathbf{1}_B \mu$ is a Borel measure which a priori need not have a finite mass.

However we can approximate ν by an increasing sequence of non pluripolar measures of finite masses on X defined by

$$\nu_j := \mathbf{1}_{\Omega \setminus B} e^{-\max(\varphi, -j)} MA(\varphi) + \mathbf{1}_B \mu$$

and solve the corresponding equations :

$$MA(\psi_j) = \mathbf{1}_{\Omega \setminus B} e^{\psi_j - \max(\varphi, -j))} MA(\varphi) + \mathbf{1}_B e^{\psi_j} \mu = e^{\psi_j} \nu. \tag{6}$$

By the comparison principle, the sequence (ψ_j) decreases to a function $\psi \geq u_0$ since u_0 is a subsolution of these equations.

By the comparison principle, the sequence (ψ_j) decreases to a function $\psi \geq u_0$ since u_0 is a subsolution of these equations.

We claim that ψ solves the equation (5). Assume this is the case. Then observe that φ is a supersolution of the equation (5) and then by the comparison principle it follows that $\psi \leq \varphi$ in X.

On the other hand ψ is a supersolution to the equation (4) and then $\varphi \leq \psi$ by minimality. Thus $\varphi = \psi$.

By the comparison principle, the sequence (ψ_j) decreases to a function $\psi \geq u_0$ since u_0 is a subsolution of these equations.

We claim that ψ solves the equation (5). Assume this is the case. Then observe that φ is a supersolution of the equation (5) and then by the comparison principle it follows that $\psi \leq \varphi$ in X.

On the other hand ψ is a supersolution to the equation (4) and then $\varphi \leq \psi$ by minimality. Thus $\varphi = \psi$.

It remains to prove that ψ is a solution to the equation (5). The inequality $\geq$ follows from Fatou's lemma and equality follows from the fact that the masses coincide.

Kähler-Einstein metrics

Let us give a geometric situation where this theorem applies.
Let Y be an algebraic variety with mild singularities. We assume that the canonical bundle $K_Y := \wedge^n (T^{1,0}X)^*$ is ample meaning that it has a hermitian metric with positive Ricci curvature i.e. there exists a Kähler metric ω_Y on Y such that $\omega_Y \in c_1(K_Y) = -c_1(Y)$.

Kähler-Einstein metrics

Let us give a geometric situation where this theorem applies.
Let Y be an algebraic variety with mild singularities. We assume that the canonical bundle $K_Y := \wedge^n (T^{1,0}X)^*$ is ample meaning that it has a hermitian metric with positive Ricci curvature i.e. there exists a Kähler metric ω_Y on Y such that $\omega_Y \in c_1(K_Y) = -c_1(Y)$.

We look for a Kähler-Einstein metric $\tilde{\omega}$ on Y i.e. $\mathrm{Ric}\,(\tilde{\omega}) = -\tilde{\omega}$.

Let $\pi : X \longrightarrow Y$ a smooth resolution of Y and let $\theta := \pi^*(\omega_Y)$. Then the Kähler-Einstein equation $\mathrm{Ric}\,(\tilde{\omega}) = -\tilde{\omega}$ on Y is equivalent to a complex Monge-Ampère on X

$$MA_\theta(\varphi) = e^\varphi \mu,$$

where $\mu := fdV$, and $f \geq 0$ is a density which is smooth on $X \setminus D := \pi^{-1}(Y_{reg})$ and may vanish or blow up along the divisor $D := \pi^{-1}(Y_{sing})$.

The way how f blows up depends on the type of singularities of Y. The case of klt singularities corresponds to the case when $f \in L^{1+\varepsilon}(X)$ and has been studied in [EGZ09,BBEGZ10].

The way how f blows up depends on the type of singularities of Y. The case of klt singularities corresponds to the case when $f \in L^{1+\varepsilon}(X)$ and has been studied in [EGZ09,BBEGZ10].

The most delicate case is when Y has semi-log canonical singularties i.e. $f \leq C|s_D|^{-2}$, where s_D is a defining holomorphic section for D i.e. $D = \{s_D = 0\}$. This case was recently studied by Guenancia and Berman. Let us see how to apply our theorem to solve the above corresponding equation.

The way how f blows up depends on the type of singularities of Y. The case of klt singularities corresponds to the case when $f \in L^{1+\varepsilon}(X)$ and has been studied in [EGZ09,BBEGZ10].

The most delicate case is when Y has semi-log canonical singularties i.e. $f \leq C|s_D|^{-2}$, where s_D is a defining holomorphic section for D i.e. $D = \{s_D = 0\}$. This case was recently studied by Guenancia and Berman. Let us see how to apply our theorem to solve the above corresponding equation.

Here $f \notin L^1(X)$ but we have a subsolution of the equation : take $u_0 := -(-\log|s_D|^{-2})^a$, where $0 < a < 1$ and s_D is appropriately normalized.

Then we can apply our theorem to obtain

Corollary

Let Y be an algebraic variety with semi-log canonical singularties such that K_Y is ample. Then there exists a Kähler metric ω_{KE} on Y_{reg} such that $\mathrm{Ric}\omega_{KE} = -\omega_{KE}$. Moreover the current ω_{KE} extends as a positive closed current on X with finite energy potential on X.

References

- [Aub78] T. Aubin: *Equation de type Monge-Ampère sur les variétés kählériennes compactes.* Bull. Sci. Math. **102** (1978), 63–95.
- [BT76] E. Bedford, B.A. Taylor : *The Dirichlet problem for a complex Monge-Ampère equation*, Invent. Math. **37** (1976), no. 1, 1–44.
- [BT82] E. Bedford, B.A. Taylor : *A new capacity for plurisubharmonic functions*, Acta Math. **149** (1982), 1–40.
- [BT87] E. Bedford, B.A. Taylor : *Fine topology, Šilov boundary, and* $(dd^c)^n$, J. Funct. Anal. 72 (1987), no. 2, 225–251.
- [Ber13] R.J. Berman, *From Monge-Ampère equations to envelopes and geodesic rays in the zero temperature limit*, https://arxiv.org/abs/1307.3008arXiv:1307.3008.
- [Ber13] R.J. Berman, S. Boucksom : *Growth of balls of holomorphic sections and energy at equilibrium*, arXiv:0803.1950
- [BEGZ10] S. Boucksom, P. Eyssidieux, V. Guedj, A. Zeriahi, *Monge-Ampère equations in big cohomology classes*, Acta Math.

- [BG13] R.J. Berman, H. Guenancia : *Kähler-Einstein metrics on stable varieties and log canonical pairs*, arXiv:1304.2087v2
- [CIL92] M. Crandall, H. Ishii, P.L. Lions : *User's guide to viscosity solutions of second order partial differential equations* Bull. Amer. Math. Soc. **27** (1992), 1–67.
- [Dar14] T. Darvas : The Mabuchi completion of the space of Kähler potentials, Amer. J. Math. 139 (2017), no. 5, 1275-1313. arXiv:1401.7318
- [DR16] T. Darvas, Y. Rubinstein : *Kiselman's principle, the Dirichlet problem for the Monge-Ampre equation, and rooftop obstacle problems*. J. Math. Soc. Japan 68 (2016), no. 2, 773796.
- [EGZ09] P. Eyssidieux, V. Guedj, A. Zeriahi : *Singular Kähler-Einstein metrics*, J. Amer. Math. Soc. **22** (2009), 607–639.
- [EGZ11] P. Eyssidieux, V. Guedj, A. Zeriahi : *Viscosity solutions to Degenerate Complex Monge-Ampère Equations*, Comm. Pure Appl. Math. **64** (2011), no. 8, 1059–1094.
- [GZ07] V. Guedj, A. Zeriahi : *The weighted Monge-Ampère energy of quasiplurisubharmonic functions*, J. Funct. Anal. **250**

- [GZbook] V. Guedj, A. Zeriahi : *Degenerate Complex Monge-Ampère Equations*. EMS Tracts in Mathematics, Vol. 26, 2017.
- [HL09] F.R. Harvey, H.B. Lawson : *Dirichlet duality and the nonlinear Dirichlet problem*, Comm. Pure Appl. Math. **62** (2009), no. 3, 396–443.
- [Kol98] S. Kolodziej : *The complex Monge-Ampère equation*, Acta Math. **180** (1998), no. 1, 69–117.
- [Kol03] S. Kolodziej : *The Monge-Ampère equation on compact Kähler manifolds*, Indiana Univ. Math. J. **52** (2003), 667–686.
- J. Roos, D. Witt-Nyström : *Envelopes with prescribed singularities*, arXiv:1210.2220v2
- [Yau78] S.T. Yau :*On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation*, Comm. Pure Appl. Math. **31**, 339–441 (1978).

Thank you for your attention

and

Good luck for SMMER