Large values of class numbers of real quadratic fields

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Colloque mathématique en l'honneur des mathématiciens marocains à l'étranger
Kénitra, Septembre 23, 2016

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- $h(K)$ is a measure of how badly factorization in the ring of integers $\mathcal{O}_K$ fails to be unique.

Quadratic fields

Notation

- A quadratic field is a field of degree 2 over the rationals.
- These are the fields $\mathbb{Q}(\sqrt{d})$, where d is a fundamental discriminant ($d \equiv 1 \bmod 4$ and is a square-free integer, or $d = 4m$ with $m \equiv 2$ or $3 \bmod 4$ and m is a square-free integer).

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- The number of imaginary quadratic fields with a given class number h is **finite**.
- There are infinitely many real quadratic fields with class number 1.

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Baker (1966), Heegner (1952) and Stark (1967)

There are exactly **nine** imaginary quadratic fields with class number 1, namely those corresponding to discriminants:

$$-3, -4, -7, -8, -11, -19, -43, -67, -163.$$

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Watkins (2004)

Determined the list of all imaginary quadratic fields with class number $h \leq 100$.

Why is it easier for imaginary vs real quadratic fields?

Dirichlet's class number formula (1839)

If $d < 0$ is a fundamental discriminant, then

$$h(d) = \frac{\omega}{2\pi}\sqrt{|d|} \cdot L(1, \chi_d),$$

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- The analogous class number formula for real quadratic fields is more complicated, due to the appearance of **non-trivial units** in this case.

Dirichlet characters and Dirichlet L-functions

A Dirichlet character modulo q is a function $\chi : \mathbb{Z} \to \mathbb{C}$ such that

1. χ is periodic with period q.
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$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \text{ for } \mathrm{Re}(s) > 1.$$

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- $L(s, \chi)$ can be analytically continued to an entire function over the complex plane $\mathbb{C}$.
- $L(s, \chi)$ has a functional equation which relates $L(s, \chi)$ to $L(1 - s, \overline{\chi})$.

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The Generalized Riemann Hypothesis GRH (1,000,000 $)

Let χ be a Dirichlet character modulo q. If ρ is a complex number with $0 \leq \mathrm{Re}(\rho) \leq 1$ and $L(\rho, \chi) = 0$, then $\mathrm{Re}(\rho) = \frac{1}{2}$.

How small (or large) can $L(1, \chi_d)$ be?

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Siegel's Theorem (1931)

- For every $\epsilon > 0$, there exists a positive constant c_ϵ such that

$$L(1, \chi_d) \geq c_\epsilon |d|^{-\epsilon}.$$

- The proof is not effective.

Conditional bounds for $L(1, \chi_d)$

Theorem 1 (Littlewood, 1928)

Assume the Generalized Riemann Hypothesis GRH. Then

$$(\zeta(2) + o(1))(2e^{\gamma} \log\log |d|)^{-1} \leq L(1, \chi_d) \leq (2e^{\gamma} + o(1)) \log\log |d|,$$

where γ is the Euler-Mascheroni constant, and $\zeta(2) = \pi^2/6$.

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Assume GRH. If $d < -4$ is a fundamental discriminant, then

$$\left(\frac{\zeta(2)}{2\pi e^{\gamma}} + o(1)\right)\frac{\sqrt{|d|}}{\log\log|d|} \leq h(d) \leq \left(\frac{2e^{\gamma}}{\pi} + o(1)\right)\sqrt{|d|}\cdot\log\log|d|.$$

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Conjecture (Gauss, 1801)

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Theorem (Montgomery and Weinberger, 1977)

There exists a positive constant c, and infinitely many real quadratic fields $\mathbb{Q}(\sqrt{d})$ such that

$$h(d) \geq c \cdot \sqrt{d} \cdot \frac{\log\log d}{\log d}.$$

Conjecture (Montgomery and Vaughan, 1999)

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(b) Furthermore, there are **at most** $x^{1/2+o(1)}$ real quadratic fields $\mathbb{Q}(\sqrt{d})$ with discriminant $d \leq x$, for which (1) holds.

Chowla's family of real quadratic fields

- To prove part a) we use the following family of fundamental discriminants, first studied by Chowla:

$$\mathcal{D}_{\text{ch}} := \{d : d \text{ squarefree of the form } d = 4m^2 + 1 \text{ for } m \geq 1\}.$$

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- If $d = 4m^2 + 1$ is square-free then $\varepsilon_d = 2m + \sqrt{d} = \sqrt{d-1} + \sqrt{d}$.
- By Dirichlet's class number formula, if $d \in \mathcal{D}_{\text{ch}}$ then

$$h(d) = \frac{\sqrt{d}}{\log(\sqrt{d-1} + \sqrt{d})} L(1, \chi_d).$$

Class number 1 problem for Chowla's family

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Theorem (Biró, 2003)

Chowla's conjecture is true.

The number of fields with a given class number: Case I imaginary quadratic

$\mathcal{F}_{\mathrm{im}}(h) = |\{\text{Imaginary quadratic fields } \mathbb{Q}(\sqrt{d}) : h(d) = h\}|.$

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Conjecture (Soundararajan, 2007)

$$c_1 \cdot \frac{h}{\log h} \leq \mathcal{F}_{\text{im}}(h) \leq c_2 \cdot h \log h,$$

for some positive constants c_1, c_2, if h is sufficiently large.

Conjecture (Holmin, Jones, Kurlberg, McLeman, Petersen, 2015)

As $h \to \infty$ through odd values we have

$$\mathcal{F}_{\text{im}}(h) \sim \mathcal{C} \cdot c(h) \cdot \frac{h}{\log h}$$

where

$$\mathcal{C} = 15 \prod_{p>2} \prod_{i=2}^{\infty} \left(1 - \frac{1}{p^i}\right) \approx 11.317 \text{ and } c(h) = \prod_{p^n \| h} \prod_{i=1}^{n} \left(1 - \frac{1}{p^i}\right)^{-1}.$$

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Theorem (Soundararajan, 2007)

For large h we have

$$\mathcal{F}_{\text{im}}(h) = O\left(h^2 \frac{(\log \log h)^4}{(\log h)^4}\right).$$

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Theorem (L, 2015)

$$\sum_{h \le H} \mathcal{F}_{\mathrm{im}}(h) = \frac{3\zeta(2)}{\zeta(3)} H^2 + O\left(\frac{H^2(\log\log H)^3}{\log H}\right).$$

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Theorem (Dahl and L, 2016)

$$\sum_{h \le H} \mathcal{F}_{\text{ch}}(h) = \frac{1}{2G} H \log H + O\left(H(\log\log H)^3\right),$$

where

$$G = L(2, \chi_{-4}) = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} + \cdots = 0.916\ldots$$

is Catalan's constant, and χ_{-4} is the non-principal character modulo 4.

The key idea: A probabilistic random model for $h(d)$

Recall that

$$h(d) = \frac{\sqrt{d}}{\log(\sqrt{d-1} + \sqrt{d})} \cdot L(1, \chi_d),$$

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- We expect the values $\chi_d(p)$ to behave (essentially) **independently** for different primes p.

Strategy

- "Construct a random Euler product"

$$L(1, \mathbb{X}) := \prod_p \left(1 - \frac{\mathbb{X}(p)}{p}\right)^{-1},$$

where $\mathbb{X}(p)$ are independent random variables taking the values $1, -1$ and 0 with the probabilities α_p, β_p and γ_p respectively.

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- We show that **the complex moments** of $L(1, \chi_d)$ as d varies in $\mathcal{D}_{\text{ch}}$ are nearly equal to those of the random model $L(1, \mathbb{X})$.

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- We show that **the complex moments** of $L(1, \chi_d)$ as d varies in $\mathcal{D}_{\text{ch}}$ are nearly equal to those of the random model $L(1, \mathbb{X})$.
- We use **the saddle-point method** to prove that the distribution of $L(1, \chi_d)$ as d varies in $\mathcal{D}_{\text{ch}}$ is very close to that of the probabilistic model $L(1, \mathbb{X})$, in a large range.

Thank you for your attention !