# The amenability to algebraic and analytical perspective and some contributions

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This presentation is a synthesis on my contributions that I have made on the **subject of the amenability** to algebraic and analytical perspective since 2003.

1. Contractible Frèchet algebras. Proceedings of the American Mathematical Society. Vol 132, Number 5 Pages : 1251-1255 (2003).

2. The structure of a subclass of amenable Banach algebras, Int. J. Math. Math. Sc, Volume 2004, 55 ( 2004) 2963-2969.

3. Reduction operator algebras and Generalized Similarity Problem, Operators and Matrices, Volume 4, Number 4, 559- 572 (2010).

4. The semisimplicity of amenable operator algebras. Archiv der Mathematik August 2013, Volume 101, Issue 2, pp 129-133. ( With Paulo Pinto)

5. Amenable Cross product Banach algebras associated with a class of dynamical systems. Integral Equation and Operator Theory. 2016 ( with Marcel de Jeu)

# Banach algebras

→ A Banach algebra $A$ is a Banach space provided a compatible product with other operations and the norm $\|.\|$ satisfying for all $a, b \in A$,

$$\|a.b\| \leq \|a\|\|b\|.$$

- $C(X)$ with $X$ a compact set. $\|f\|_\infty = sup\{|f(x)|, x \in X\}$.
- $B(H)$, $K(H)$ with $H$ a Hilbert space. $\|T\| = sup\{\|T(x)\|, \|x\| = 1, x \in H\}$
- For each $T \in B(H)$, Let $A_T = \overline{span\{P(T), P \in \mathbb{C}[X]\}}^{\|\ \|}$.
- A disc algebra $\mathbb{A}(D)$.

→ A C$^*$-algebra $(A, \|.\|, *)$ is a Banach algebra with an involution such that for all $a \in A$,

$$\|a.a^*\| = \|a\|^2.$$

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# Operator algebras

$\rightarrow A$ is called an operator algebra if $A \subseteq B(H)$ is an algebra, closed under the norm topology, for some Hilbert space $H$.

For a locally compact group $G$, a unitary representation $\pi$ on some Hilbert space $H$ is continuous homomorphism into the unitary group $\mathcal{U}$ of $B(H)$.

$\lambda : G \longrightarrow B(L^2(G)),\ \ (\lambda(t)(f))(h) = f(h^{-1}t)$ is called left regular representation.

$\tilde{\pi} : L^1(G) \longrightarrow B(H)\ \ \tilde{\pi}(f) = \int_G f(t)\pi(t)d\mu(t)\ \ f \in L^1(G)$.

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# More examples

• $C_b(G)$, Big algebra Big$(G)$, Group von Neumann algebra $VN(G)$, Measure algebra $M(G)$, The Fourier-Steiljes algebra

$$B(G) = \{u \in C_b(G) : u =< \pi(.)\xi, \eta > \ for\ some\ \pi \in \hat{G},\ \xi,\ \eta \in H_\pi\}.$$

• The Fourier algebra

$$A(G) = \overline{\{u =< \lambda(.)\xi, \eta >,\ \lambda\ left\ regular\ representation,\ \xi, \eta \in L^2(G)\}}^{\|\ \|_{B(G)}}.$$

$$\|u\|_{A(G)} := \|u\|_{B(G)} = inf\{\|\xi\|_2\|\eta\|_2 : \xi,\ \eta \in L^2(G)\}.$$

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**Definition**

Let $A$ be a (Banach ) complex algebra. $A$ is called contractible if for any bimodule $X$ on $A$, every derivation $D : A \to X$ is inner.

**Theorem**

$A$ is contratible iff $A$ is semisimple finite dimensional algebra.

**Definition**

Let $A$ be a Banach complex algebra. $A$ is called **syper amenable** if for any Banach bimodule $X$ on $A$, every bounded derivation $D : A \to X$ is inner.

**Theorem A. Ya. Helemskii**

$A$ is commutative syper amenable iff $A$ is semisimple finite dimensional algebra.

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- No infinte dimesional exemple until yet.
- Helemskii 's Conjecture : Syper amenable Banach algebra is finte dimensional.

**Theorem El Harti IJMMS 2004**

Let $A$ be an syper amenable Banach algebra such that each left maximal ideal is complemented. then $A$ is semisimple finite dimensional algebra.

**1. A syper amenable C*-algebra is finite dimensional algebra.**

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**3. A syper amenable Hermitian involutive Banach algebra is finite dimensional algebra.**

**Definition**

Let $A$ be a Banach complex algebra. $A$ is called **amenable** if for any bimodule $X$ on $A$, every derivation $D : A \to X^*$ into the dual of $X$ is inner.

# Algebraic definition and super amenability

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Let $A$ be a Banach complex algebra. $A$ is called **amenable** if for any bimodule $X$ on $A$, every derivation $D : A \to X^*$ into the dual of $X$ is inner.

# Amenable groups

By John von Neumann in 1929

**Definition**

An amenable group $G$ is a topological group with a lefy invariant mean on the algebra $C_{ru}(G)$ of the right uniformely continous functions on $G$.

- Examples : Compact groups, Abelien groups

The Heisenberg group $\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, a, b, c \in \}$

- $0 \longrightarrow N \longrightarrow G \longrightarrow G/N \longrightarrow 0.$

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Any discrete group contains free group as a copy.

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# Amenable Banach algebras

**Theorem B.E. Johnson 1971**

A locally compact group $G$ is amenable if and only if for any bimodule $X$ on $L^1(G)$, every derivation $D : L^1(G) \longrightarrow X^*$ is inner where $X^*$ is the dual of $X$

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A Banach algebra is amenable if for any bimodule $X$ on $A$, every derivation $D : A \longrightarrow X^*$ is inner.

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Examples : $K(H)$, $C(X)$, Cuntz algebras $O_n$, $L^1(G)$. $C^*(G)$ if $G$ amenable, $C^*(Sl_2(R))$.

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# Amenable Group Algebras

**Theorem**

Let $G$ be a loccaly compact group. Then the following are equivalent :
1. $G$ is amenable.
2. $L^1(G)$ is amenable.
3. $A(G)$ has a bounded approximate identity
4. $B(G)$ has an identity.
5. $C^*(G) = C_r^*(G)$ and amenable.

A bounded approximate identity is a bounded net $\{e_\alpha\}_{\alpha \in I}$ in $A$ with that $\|e_\alpha\| \leq K$ for some real $K$ and

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Note : Every closed tow-sided ideal $J$ with a bounded approximate identity in an amenable Banach algebra $A$ is amenable.

**Theorem R. El Harti 2010. J.O.M**

Every closed two sided ideal $J$ of an amenable operator algebra has a bounded approximate identity.

- So, Every closed tow-sided ideal $J$ of an amenable operator algebra $A$ is amenable.

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# Finite dimensional case

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Let $A$ be a finite dimensional algebra, Then the following are equivalent :
1. $A$ is amenable.
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3. $A$ has a diagonal. $\ni d \in A \otimes A$ such that $ad = da$ for each $a \in A$ and $\pi(d) = 1$ where $\pi; A \otimes A \to A$.
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$$A \equiv \mathbb{M}_{n_1}(\mathbb{C}) \oplus \mathbb{M}_{n_1}(\mathbb{C} \oplus ....... \oplus \mathbb{M}_{n_k}(\mathbb{C}$$
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# Problems

Natural Question : If $dim(A) = +\infty$

$$Amenability \leftrightarrows Smisimplicity$$

Answer : No

$\leftarrow B(H)$ with $dim(H) = +\infty$ semisimple but not amenable.

$\rightarrow$ By Read $\exists$ a radical amenable Banach algebra wich is not an operator algebra.

- Is an amenable operator algebra semisimple ?

**Definition**

A Banach algebra $A$ satisfies to Wedderburn property (W) if for each a closed two sided ideal $I$ which is complemented as a Banach space. There is a closed two sided ideal $J$ such that $A = I \oplus J$.

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# Main theorem

**Theorem : R. El Harti and P. Pinto. Arckiv Math. July 2013**

Let $A$ be a reflexive amenable operator algebra. Then it is semisimple with property (W). In this case, it is a finite direct sum of simple Banach algebras of operators.

Proof : First show that every closed ideal of an operator algebra has b.a.i.
Let $\pi : A \to B(H)$ be a bounded representation of $A$ on some Hilbert space $H$. Let $M$ be a closed invariant subspace of $\pi(A)$ and take the following admissible short sequence

$$0 \longrightarrow M \longrightarrow H \to H/M \longrightarrow 0.$$

By [Curtis], this sequence splits, therefore $A$ has the total reduction property. It follows from [RElHarti] that every closed two-sided ideal of $A$ has a bounded approximate identity. Therefore the result is now an easy consequence of the following results .

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**Lemma 1.**

Let $A$ be a reflexive operator algebra such that every maximal two sided ideal of $A$ has a bounded approximate identity. Then every primitive ideal of $A$ is maximal.

Proof : Let $P$ be a primitive ideal. Then $B := A/P$ is a primitive operator algebra. Is $B$ is simple ? For a maximal two-sided ideal $M_B$ in $B$. Then $M_B$ has a bounded approximate identity (since $M_B = (M_A + P)/P$ for some maximal two-sided ideal $M_A$ in $A$).
Then $\overline{M_B}^{w^*}$ is a two-sided ideal in $B^{**}$ and $\overline{M_B}^{w^*} = B^{**}p$ with $p \in B^{**}$ some central idempotent [Effros]. Besides this,

$$B^{**} = B^{**}p \oplus B^{**}(1_{B^{**}} - p), \tag{1}$$

Since the reflexivity property passes to quotients we have that $B$ is also reflexive. Thus from (1) we conclude that $B = Bp \oplus B(1 - p)$ with $Bp$ and $B(1 - p)$ being two-sided ideals in $B$. However every non-trivial two-sided ideal in the primitive algebra $B$ is essential (an ideal $I$ is said to be essential if $I \cap J$ is non-trivial for all non-trivial ideal $J$. It follows that $Bp = \{0\}$ or $B(1 - p) = \{0\}$. Since $M_B = Bp \neq B$ and $p \neq 1$ we conclude that $Bp = \{0\}$. Hence $M_B = \{0\}$ and so $B$ is simple.

**Lemma 1.**

Let $A$ be a reflexive operator algebra such that every maximal two sided ideal of $A$ has a bounded approximate identity. Then every primitive ideal of $A$ is maximal.

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# Main results

**Lemma 2.**

Let $A$ be a reflexive operator algebra such that each maximal two sided ideal has a bounded approximate identity. Then $A$ is semisimple. Moreover, $A$ is a finite direct sum of simple operator algebras.

Proof : Let $\Pi_A$ be the space of all primitive ideals in $A$ equipped with the hull kernel topology. If $P \in \Pi_A$, then $P$ is maximal by Lemma. Therefore $P$ has a bounded approximate identity and $P^{**} = A^{**}p$ for some central idempotent $p$ by [Effros]. Since $A$ is reflexive, $P = Ap$. Using the same argument in [Galé Ransford, White], we conclude that $\Pi_A$ is discrete and compact. Hence $\Pi_A$ is a finite set, say $\Pi_A = \{P_1, \ldots, P_n\}$ with central idempotents $p_1, \ldots, p_n$, respectively. It is easy to check that

$$A = Ap_1p_2\ldots p_n \oplus \bigoplus_{i=1}^{n} A(1-p_i), \qquad \mathrm{Rad}(A) = Ap_1p_2\ldots p_n = \bigcap_{i=1}^{n} Ap_i.$$

Therefore Rad$(A) = \{0\}$ and $A(1-p_i)$ is a minimal two sided ideal (for every $i = 1, \ldots, n$). Thus $A$ is semisimple and moreover $A$ is a finite direct sum of simple algebras.

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# Main result : Commutative case

**Theorem. (Rachid,Pnto)**

If $A_T^0$ amenable and contains a non trivial compact operator $K$, then $T$ is non quasinilpotent.

Proof : • We show that $TK \neq 0$. Indeed, if that is not the case, then since $K \in A_T^0$, $K$ is a limit of polynomial $P_n(T)$ with $P_n(0) = 0$. So $K^2$ is the limit of $P_n(T)K$. Note now that $P_n(T)K = 0$ for all $n$, so $K^2 = 0$ thus $K$ is nilpotent. This implies that $A_K^0$ is finite dimensional amenable algebra and thus it is semisimple algebra. Therefore $K = 0$

• So since $TK \neq 0$, there exists a trace-class operator $N \in C(H)$ such that tr$(TKN) \neq 0$. Let $D_N$ be the derivation from $A_T^0$ to $(A_{TK}^0)^{\top}$ defined by $D_N(A) := NA - AN$ for all $A \in A_T^0 \subseteq B(H) = C(H)^*$, where $(A_{TK}^0)^{\top}$ is the annihilator of $A_{TK}^0$ taken in $C(H)$ (note that $A_{TK}^0 \subseteq A_T^0$, so $D : A_T^0 \to (A_T^0)^{\top} \subseteq (A_{TK}^0)^{\top}$). Besides this, $(A_{TK}^0)^{\top}$ is a Banach $A_T^0$-bimodule in $C(H)$. Since $A_T^0$ is amenable $D_N$ is inner, so there exists an $M \in (A_{TK}^0)^{\top}$ such that $D_N(A) = MA - AM$ for all $A \in A_T^0$. This means that $KT(N - M) = (N - M)KT$ and tr$(KT(N - M)) =$ tr$((N - M)KT) \neq 0$. Hence $\sigma(KT(N - M)) \neq \{0\}$ where $\sigma(KT(N - M))$ denotes the spectrum of $KT(N - M)$. Since $KT$ and $N - M$ commute we have $\sigma(KT(N - M)) \subseteq \sigma(KT)\sigma(N - M)$ and therefore $\sigma(KT) \neq \{0\}$. Similarly, $\sigma(KT) \subseteq \sigma(K)\sigma(T)$, whence $\sigma(T) \neq \{0\}$. Therefore $T$ is non quasinilpotent.

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# Amenable Cross product Banach algebras associated with a class of dynamical systems.

Let $(G, \alpha, A)$ a discret C*dynamical System.

$$l^1(G,\alpha,A) = \{a : G \longrightarrow A \ such\ that\ \sum_{g\in G} \|a(g)\|_A < \infty\}.$$

We supply $l^1(G,\alpha,A)$ with

$$(ab)_g = \sum_{t\in G} a_t.\alpha_t(b_{t^{-1}g}) \quad (g \in G,\ a,\ b\ \in l^1(G,\alpha,A))$$

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$$a_g^* = \alpha_g((a_{g^{-1}})^*).$$

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Two Examples :

1. $A = \mathbb{C}$, then $l^1(G, triv, C)$ is the usual group algebra $l^1(G)$.

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$\delta_g \in l^1(G, \alpha, A)$ and $\delta_e$ is the identity of $l^1(G, \alpha, A)$.

Each element $a = (a_g)_{g \in G}$ of $l^1(G, \alpha, A)$ can be written in the form

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$$\alpha_g(a) = \delta_g a \delta_{g^{-1}} \qquad g \in G \quad and\ a \in A$$

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**Lemma .**

Let $(G, \alpha, A)$ be a C*-dynamical system where A is unital and G is discrete. The the set $\{u\delta_g, u \in Uandg \in G\}$ is a subgroup of invertible elements of $l^1(G, \alpha, A)$ that is canonically isomorphic to the semidirect product group $U \ltimes_\alpha G$. The norm closed linear space of this set $l^1(G, \alpha, A)$.

# Amenable Cross product Banach algebras associated with a class of dynamical systems.

**Theorem : R. El Harti and Marcel de Jeu July 2016**

Let $(G, \alpha, A)$ be a C*-dynamical system where A is commutative unital C*-algebra and G is amenable discrete group. Then $l^1(G, \alpha, A)$ is amenable.

Proof :

1. step

- With $G$ amenable and $U$ is abelian group, we check that $U \ltimes_\alpha G$ is amenable.

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Merci bien pour votre attention ]