The amenability to algebraic and analytical perspective and some contributions

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Universitát des Saarlandes, Saarbrúcken Germany 12 / 12/ 2016



This presentation is a synthesis on my contributions that I have made on the **subject of the amenability** to algebraic and analytical perspective since 2003.

- 1. Contractible Frèchet algebras. Proceedings of the American Mathematical Society. Vol 132, Number 5 Pages: 1251-1255 (2003).
- 2. The structure of a subclass of amenable Banach algebras, Int. J. Math. Math. Sc, Volume 2004, 55 (2004) 2963-2969.
- 3. Reduction operator algebras and Generalized Similarity Problem, Operators and Matrices, Volume 4, Number 4, 559-572 (2010).
- 4. The semisimplicity of amenable operator algebras. Archiv der Mathematik August 2013, Volume 101, Issue 2, pp 129-133. (With Paulo Pinto)
- 5. Amenable Cross product Banach algebras associated with a class of dynamical systems. Integral Equation and Operator Theory. 2016 (with Marcel de Jeu)

$$||a.b|| \le ||a|| ||b||.$$

- C(X) with X a compact set. $||f||_{\infty} = \sup\{|f(x)|, x \in X\}.$
- B(H), K(H) with H a Hilbert space.

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- For each $T \in B(H)$, Let $A_T = \overline{span\{P(T), P \in \mathbb{C}[X]\}^{\|\cdot\|}}$.
- A disc algebra A(D).
- \rightarrow A C*-algebra $(A, \|.\|, *)$ is a Banach algebra with an involution such that for all $a \in A$,

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For a locally compact group G, a unitary representation π on some Hilbert space H is continuous homomorphism into the unitary group \mathcal{U} of B(H).

 $\lambda: G \longrightarrow B(L^2(G)), \ \ (\lambda(t)(f))(h) = f(h^{-1}t)$ is called left regular representation.

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Let A be a (Banach) complex algebra. A is called contractible if for any bimodule X on A, every derivation $D:A\to X$ is inner.

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A is contratible iff A is semisimple finite dimensional algebra.

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Theorem El Harti IJMMS 2004

Let *A* be an syper amenable Banach algebra such that each left maximal ideal is complemented, then *A* is semisimple finite dimensional algebra.

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Amenable groups

By John von Neumann in 1929

Definition

An amenable group G is a topological group with a lefy invariant mean on the algebra $C_{ru}(G)$ of the right uniformly continous functions on G.

Examples: Compact groups, Abelien groups

The Heisenberg group
$$\left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, a,b,c \in \right\}$$

- $\bullet 0 \longrightarrow N \longrightarrow G \longrightarrow G/N \longrightarrow 0.$
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A locally compact group G is amenable if and only if for any bimodule X on $L^1(G)$, every derivation $D:L^1(G)\longrightarrow X^*$ is inner where X^* is the dual of X

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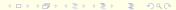


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$$A \equiv \mathbb{M}_{n_1}(\mathbb{C}) \oplus \mathbb{M}_{n_1}(\mathbb{C} \oplus \ldots \oplus \mathbb{M}_{n_k}(\mathbb{C})$$

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Answer: No

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Theorem: R. El Harti and P. Pinto. Arckiv Math. July 2013

Let *A* be a reflexive amenable operator algebra. Then it is semisimple with property (W). In this case, it is a finite direct sum of simple Banach algebras of operators.

Proof : First show that every closed ideal of an operator algebra has b.a.i. Let $\pi:A\to B(H)$ be a bounded representation of A on some Hilbert space H. Let M be a closed invariant subspace of $\pi(A)$ and take the following admissible short sequence

$$0 \longrightarrow M \longrightarrow H \rightarrow H/M \longrightarrow 0.$$

By [Curtis], this sequence splits, therefore A has the total reduction property. It follows from [REIHarti] that every closed two-sided ideal of A has a bounded approximate identity. Therefore the result is now an easy consequence of the following results .

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Lemma 1.

Let A be a reflexive operator algebra such that every maximal two sided ideal of A has a bounded approximate identity. Then every primitive ideal of A is maximal.

Proof : Let P be a primitive ideal. Then B := A/P is a primitive operator algebra. Is B is simple ? For a maximal two-sided ideal M_B in B. Then M_B has a bounded approximate identity (since $M_B = (M_A + P)/P$ for some maximal two-sided ideal M_A in A).

Then $M_B^{"}$ is a two-sided ideal in B^{**} and $M_B^{"}=B^{**}p$ with $p\in B^{**}$ some central idempotent [Effros]. Besides this,

$$B^{**} = B^{**}p \oplus B^{**}(1_{B^{**}} - p), \tag{1}$$

Since the reflexivity property passes to quotients we have that B is also reflexive. Thus from (1) we conclude that $B=Bp\oplus B(1-p)$ with Bp and B(1-p) being two-sided ideals in B. However every non-trivial two-sided ideal in the primitive algebra B is essential (an ideal I is said to be essential if $I\cap J$ is non-trivial for all non-trivial ideal J. It follows that $Bp=\{0\}$ or $B(1-p)=\{0\}$. Since $M_B=Bp\neq B$ and $p\neq 1$ we conclude that $Bp=\{0\}$. Hence $M_B=\{0\}$ and so B is simple.



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Main results

Lemma 2

Let A be a reflexive operator algebra such that each maximal two sided ideal has a bounded approximate identity. Then A is semisimple. Moreover, A is a finite direct sum of simple operator algebras.

Proof : Let Π_A be the space of all primitive ideals in A equipped with the hull kernel topology. If $P \in \Pi_A$, then P is maximal by Lemma. Therefore P has a bounded approximate identity and $P^{**} = A^{**}p$ for some central idempotent p by [Effros]. Since A is reflexive, P = Ap. Using the same argument in [Galé Ransford, White], we conclude that Π_A is discrete and compact. Hence Π_A is a finite set, say $\Pi_A = \{P_1, \ldots, P_n\}$ with central idempotents p_1, \ldots, p_n , respectively. It is easy to check that

$$A = Ap_1p_2...p_n \oplus \bigoplus_{i=1}^n A(1 - p_i), \quad Rad(A) = Ap_1p_2...p_n = \bigcap_{i=1}^n Ap_i.$$

Therefore $Rad(A) = \{0\}$ and $A(1 - p_i)$ is a minimal two sided ideal (for every i = 1, ..., n). Thus A is semisimple and moreover A is a finite direct sum of simple algebras.



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Main result: Commutative case

Theorem. (Rachid, Pnto)

If A_T^0 amenable and contains a non trivial compact operator K, then T is non quasinilpotent.

- Proof: We show that $TK \neq 0$. Indeed, if that is not the case, then since $K \in A_T^0$, K is a limit of polynomial $P_n(T)$ with $P_n(0) = 0$. So K^2 is the limit of $P_n(T)K$. Note now that $P_n(T)K = 0$ for all n, so $K^2 = 0$ thus K is nilpotent. This implies that A_K^0 is finite dimensional amenable algebra and thus it is semisimple algebra. Therefore K = 0
- So since $TK \neq 0$, there exists a trace-class operator $N \in C(H)$ such that $\operatorname{tr}(TKN) \neq 0$. Let D_N be the derivation from A_T^0 to $(A_{TK}^0)^{\mathsf{T}}$ defined by $D_N(A) := NA AN$ for all $A \in A_T^0 \subseteq B(H) = C(H)^*$, where $(A_{TK}^0)^{\mathsf{T}}$ is the annihilator of A_{TK}^0 taken in C(H) (note that $A_{TK}^0 \subseteq A_T^0$, so
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Main result: Commutative case

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If A_T^0 amenable and contains a non trivial compact operator K, then T is non quasinilpotent.

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$$l^{1}(G, \alpha, A) = \{a : G \longrightarrow A \text{ such that } \sum_{g \in G} ||a(g)||_{A} < \infty\}.$$

We supply $l^1(G, \alpha, A)$ with

$$(ab)_g = \sum_{t \in G} a_t \cdot \alpha_t(b_{t^{-1}g}) \quad (g \in G, \ a, \ b \ \in l^1(G, \alpha, A))$$

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- 1. $A = \mathbb{C}$, then $l^1(G, triv, C)$ is the usual group algebra $l^1(G)$.
- 2. $G = \mathbb{Z}$ and A = C(X) and consider $\sigma : X \longrightarrow X$ a homeomorphism. We get a \mathbb{Z} -action defined by $\alpha_n(f) = f \circ \sigma^{-n}$.

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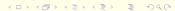
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For $g \in G$, et $\delta_g : G \longrightarrow A$ be defined by

$$\delta_g(t) = \left\{ egin{array}{ll} 1_A, & ext{if t =g;} \\ 0, & ext{if not.} \end{array}
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 $\delta_g \in l^1(G, \alpha, A)$ and δ_e is the identity of $l^1(G, \alpha, A)$.

Each element $a = (a_g)_{g \in G}$ of $l^1(G, \alpha, A)$ can be written in the form

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Lemma.

Let (G,α,A) be a C*-dynamical system where A is unital and G is discrete. The the set $\{u\delta_g, u\in Uandg\in G\}$ is a subgroup of invertible elements of $l^1(G,\alpha,A)$ that is canonically isomorphic to the semidirect product group $U\ltimes_\alpha G$. The norm closed linear space of this set $l^1(G,\alpha,A)$.

Theorem: R. El Harti and Marcel de Jeu July 2016

Let (G,α,A) be a C*-dynamical system where A is commutative unital C*-algebra and G is amenable discrete group. Then $l^1(G,\alpha,A)$ is amenable.

Proof:

1. step

• With G amenable and U is abelian group, we check that $U \ltimes_{\alpha} G$ is amenable

3. step ullet The canonical isomorphism from $Ux_{\alpha}G$ onto the set $\{u\delta_g, u\in U,\ g\in G\}$ can be extended to a homomorphism from

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with dense image.

• Since $U \ltimes_{\alpha} G$ is amenable discrete group, By Johnson $l^{1}(U \ltimes_{\alpha} G)$.

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Merci bien pour votre attention]