Some commutativity theorems in rings with involution and generalized semiderivations

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2. Definitions and notations
3. Commutativity conditions on derivations
4. Commutativity conditions on semiderivations
5. Generalized semiderivations in prime rings
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Posner's theorem

Zero is the only centralizing derivation on a non-commutative prime ring.

- it is not clear what was Posner's motivation for proving this result.

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Several authors have obtained commutativity theorems for prime and semiprime rings admitting :

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Definitions and notations

In the sequel, R will represent a 2-torsion free associative ring.

- $*$ denotes an involution of R
- $[x, y] = xy - yx \quad$ for all $x, y \in R$
- $x \circ y = xy + yx \quad$ for $x, y \in R$
- R is prime if $xRy = \{0\}$ implies $x = 0$ or $y = 0$
- R admits an involution $*$, the R is $*$-prime if $xRy = \{0\} = xRy^*$ implies $x = 0$ or $y = 0$

Note that : R prime $\Longrightarrow$ R is $*$-prime, but the converse need not be true in general.

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- An additive mapping $F : R \longrightarrow R$ is a generalized derivation if there exists a derivation d such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$.

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Commutativity conditions on derivations

Theorem 1

Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind and d a derivation of R. The following assertions are equivalent :

1. $d([x, x^*]) + [x, x^*] \in Z(R)$ for all $x \in R$;
2. $d([x, x^*]) - [x, x^*] \in Z(R)$ for all $x \in R$;
3. $d(xx^*) \pm xx^* \in Z(R)$ for all $x \in R$;
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Theorem 2

Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. Let d be a derivation of R, then the following assertions are equivalent :

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The following example proves that the primeness hypothesis in our results is not superfluous.

Example

Let $R_1 = \mathbf{Q}[X] \times T$ where T is a noncommutative 2-torsion free ring and set $d(P, t) = (P', 0)$. It is obvious that R_1 is a noncommutative ring and d is a derivation of R_1 such that $[d(r), s] = 0$ for all $r, s \in R_1$. Consider $\mathcal{R} = R_1 \times R_1^0$ provided with the involution of the second kind $*_{ex}$ given by $*_{ex}(x, y) = (y, x)$ and define $D : \mathcal{R} \longrightarrow \mathcal{R}$ by $D(x, y) = (d(x), 0)$. It is easy to verify that D is derivation of $\mathcal{R}$ which satisfies our various conditions however $\mathcal{R}$ is a noncommutative ring.

Commutativity conditions on semiderivations

Theorem 3

Let R be a 2-torsion free $*$-prime ring and J a nonzero $*$-Jordan ideal of R. If R admits a nonzero semiderivation d such that $d\Big([x,y]\Big) = 0$ for all $x, y \in J$, then R is commutative.

Bell and Daif [2, Theorem 3] showed that, if a prime ring R admits a nonzero derivation d satisfying $d\Big([x,y]\Big) = 0$ for all x, y in a nonzero ideal I of R, then R is commutative.

Commutativity conditions on semiderivations

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Let R be a 2-torsion free $*$-prime ring and J a nonzero $*$-Jordan ideal of R. If R admits a nonzero semiderivation d such that $d\Big([x,y]\Big)=0$ for all $x,y\in J$, then R is commutative.

Bell and Daif [2, Theorem 3] showed that, if a prime ring R admits a nonzero derivation d satisfying $d\Big([x,y]\Big)=0$ for all x,y in a nonzero ideal I of R, then R is commutative.

Commutativity conditions on semiderivations

Corollary 1

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Now if we consider the particular case where g is the identity map in Corollary 1, we obtain (13, Theorem 2.6).

Corollary 2 (Oukhtite-Mamouni (2012)

Let R be a 2-torsion free prime ring and J a nonzero Jordan ideal of R. If R admits a nonzero derivation d such that $d\Big([x,y]\Big)=0$ for all $x,y\in J$, then R is commutative .

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Theorem 4

Let R be a 2-torsion free $*$-prime ring and let J be a nonzero $*$-Jordan ideal of R. If R admits a semiderivation d (with associated endomorphism g) such that $d\Big([x,y]\Big) - [x,y] = 0$ for all $x, y \in J$, then R is commutative or $d(x) = x - g(x)$ for all $x \in R$.

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Application of Theorem 4 yields the following result.

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Let R be a 2-torsion free prime ring, I a nonzero ideal of R. If R admits a nonzero semiderivation d with associated function g such that $d([x,y]) - [x,y] = 0$ for all $x, y \in I$, then one of the following holds :

1. R is commutative ;
2. $d(x) = x - g(x)$ for all $x \in R$, with $g([R,R]) = 0$;
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Theorems 3 and 4 cannot be extended to semiprime rings.

Example

Let (R, σ) be a noncommutative prime ring with involution. If we set $\mathcal{R} = R \times Q[X]$, then $\mathcal{R}$ is semiprime and $(r, P(X))^* = (\sigma(r), P(X))$ is an involution of $\mathcal{R}$.

$J = \{0\} \times Q[X]$ is a $*$-Jordan ideal of $\mathcal{R}$ and $D(r, P(X)) = (0, P'(X))$ is a nonzero semiderivation of $\mathcal{R}$ associated with identity such that

$$D\Big([u, v]\Big) = 0, \ D\Big([u, v]\Big) = [u, v] \quad \text{for all} \quad u, v \in J$$

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The hypothesis "J a $*$-Jordan ideal" in Theorem 3 is crucial.

Example

Let R be a noncommutative prime ring which admits a nonzero derivation d, and let $\mathcal{R} = R \times R^0$.

If we set $J = R \times \{0\}$, then J is a nonzero Jordan ideal of the $*_{ex}$-prime ring $\mathcal{R}$.

Furthermore, if we set $D(x, y) = (0, d(y))$, then D is a semiderivation of $\mathcal{R}$ associated with identity which satisfies

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Definition (Oukhtite-Vincenzo-Mamouni)

Let $F : R \longrightarrow R$ be an additive map. If there is a semiderivation $d : R \longrightarrow R$ associated with the function $g : R \longrightarrow R$ such that

$$F(xy) = F(x)y + g(x)d(y) = d(x)g(y) + xF(y) \quad (1)$$

and

$$F(g(x)) = g(F(x)) \quad (2)$$

for each $x, y \in R$, then F is called a generalized semiderivation of R, associated with the function g and the semiderivation d.

Generalized semiderivations in prime rings

- any semiderivation is a generalized semiderivation.

- If g is the identity map of R, then all generalized semiderivations associated with g are merely generalized derivations of R.

- If R is prime and $d \neq 0$, then g must be a ring endomorphism.

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Example (Oukhtite-Vincenzo-Mamouni)

Let $d : R \longrightarrow R$ be a semiderivation of R associated with a function g of R. Define $F : R \longrightarrow R$ and $G : R \longrightarrow R$ as follows :

$$F(x) = d(x) - x, \qquad G(x) = d(x) + x, \quad \forall x \in R.$$

F and G are generalized semiderivations of R associated with g.

Our aim here is to show that any generalized semiderivation of a prime ring R assumes essentially one of the following forms :

Theorem 7

Let R be a prime ring, $F : R \longrightarrow R$ a generalized semiderivation of R associated with the endomorphism g and semiderivation d of R. Then either one of the following two cases holds :

1. The endomorphism g is the identity map of R and F is a generalized derivation of R ;
2. There exist $\alpha, \beta \in C$, the extended centroid of R, such that $F(x) = \alpha x + \beta g(x) = (\alpha + \beta)x + d(x)$, for all $x \in R$.

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We conclude this part with an application of previous results :

Theorem 8

Let R be a non-commutative prime ring of characteristic different from 2, I a non-zero ideal of R, $F : R \to R$ a non-zero generalized semiderivation associated with the endomorphism g and semiderivation d of R.

If $F([x, y]) = 0$ for all $x, y \in I$, then one of the following holds :

1. $g([R, R]) = (0)$ and there exists $\beta \in C$ such that $F(x) = \beta g(x)$ for all $x \in R$;
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Thank you