Functions spaces and related operators on Kepler Manifold

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Definitions

The Kepler manifold :

$$\mathbb{H} := \{z \in \mathbb{C}^{n+1} : z \bullet z = 0,\ z \neq 0\},$$

where $z \bullet w := z_1 w_1 + \cdots + z_{n+1} w_{n+1}$. This is the orbit of the vector $e = (1, i, 0, \ldots, 0)$ under the $O(n+1, \mathbb{C})$-action on $\mathbb{C}^{n+1}$
The manifold $\mathbb{M}$:

$$\mathbb{M} := \{z \in \mathbb{H} : |z| < 1\}.$$

The Boundary of $\mathbb{M}$:

$$\partial\mathbb{M} := \{z \in \mathbb{H} : |z| = 1\}.$$

is the orbit of the vector $\frac{\sqrt{2}}{2} e$ under the $O(n+1, \mathbb{R})$-action on $\mathbb{R}^{n+1}$ This induces a unique $O(n+1, \mathbb{R})$-invariant probability measure $d\mu$ on $\partial\mathbb{M}$.

Definitions

The holomorphic n-form α:

$$\alpha := (n+1)\frac{(-1)^{j-1}}{z_j}\, dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_{n+1} \qquad \text{on } z_j \neq 0$$

(this is, up to constant factor, the unique $SO(n+1,\mathbb{C})$-invariant holomorphic n-form on $\mathbb{H}$, see (Oljeklaus, Pflug and Y., Ann. Inst. Fourier 1997)
The $2n-1$-volume form ω:

$$\omega(z)(V_1,\ldots,V_{2n-1}) := \alpha(z)\wedge\overline{\alpha(z)}(z,V_1,\ldots,V_{2n-1})$$

$V_1,\ldots,V_{2n-1} \in T(\partial\mathbb{M})$. Mengotti, Y. 1999
Fact:

$$d\mu = \omega/\omega(\partial\mathbb{M})$$

(where, abusing notation, we denote by ω also the measure induced by ω on $\partial\mathbb{M}$). It follows, in particular, that $d\mu$ is also invariant under complex rotations

$$z \mapsto \epsilon z, \qquad \epsilon \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}.$$

For a measure ρ on $(0, +\infty)$ let

$$A^2_{\rho\otimes\mu} := \{f \in L^2(d(\rho \otimes \mu)) : f \text{ is holomorphic on } R\mathbb{M}\},$$

$$R := \sup\{t > 0 : t \in \operatorname{supp} \rho\} = \sup\{|z| : z \in \operatorname{supp} \rho \otimes \mu\}$$

understanding that $R\mathbb{M} = \mathbb{H}$ if $R = +\infty$).
$A^2_{\rho\otimes\mu}$ is a reproducing kernel Hilbert space, its Bergman kernel $K_{\rho\otimes\mu}(x, y) \equiv K(x, y)$ on $R\mathbb{M} \times R\mathbb{M}$ satisfies

$$K(\cdot, y) \in A^2_{\rho\otimes\mu}$$

for each y,

$$K(y, x) = \overline{K(x, y)}$$

, and

$$f(z) = \int_{R\mathbb{M}} f(w)K(z, w)\, d(\rho \otimes \mu)(w) \qquad \forall f \in A^2_{\rho\otimes\mu}.$$

The moments of the measure $d\rho$

$$q_k := \int_0^{\infty} t^k \, d\rho(t)$$

the values $q_k = +\infty$ being allowed

Bergman type spaces

Theorem

For $z, w \in R\mathbb{M}$ with R as above,

$$K_{\rho\otimes\mu}(z, w) = \sum_{l=0}^{\infty} \frac{(z \bullet \overline{w})^l}{d_l},$$

with

$$d_l := \frac{q_{2l}}{N(l)} \tag{1}$$

where

$$N(l) := \binom{l+n-1}{n-1} + \binom{l+n-2}{n-1} = \frac{(2l+n-1)(l+n-2)!}{l!(n-1)!}.$$

TYZ Expansion

The asymptotics of the Bergman kernel as ρ varies in a certain way (so-called Tian-Yau-Zelditch, or TYZ, expansion)

The case domains:

Consider a (Kahler potential) smooth positive function h on a bounded domain Ω in $\mathbb{C}^n$ which vanishes at the first order on the boundary:

$$h = 0 < \|\nabla h\| \quad \text{on} \quad \partial\Omega$$

and $\omega = \frac{i}{2}\partial\overline{\partial}\log\frac{1}{h}$ is a Kähler form on Ω. $A^2_{h^l\omega^n} = A^2_{h^l \det[\partial\overline{\partial}\log\frac{1}{h}]}$. Find the Kempf distortion functions

$$\epsilon_l(z) = h(z)^l K_{h^l \det[\partial\overline{\partial}\log\frac{1}{h}]}(z, z), \tag{2}$$

where K_w the weighted Bergman kernel with respect to a weight w (and similarly for A^2_w).

Functions considered by Kempf (1989), Rawnsley (1977), Ji (1989) and Zhang (1996). These functions are of importance in the study of projective embeddings and constant scalar curvature metrics Donaldson (2001), where a prominent role is played, in particular, by their asymptotic behaviour as l tends to infinity: namely, one has

$$\epsilon_l(z) \approx l^n \sum_{j=0}^{\infty} a_j(z) l^{-j} \qquad \text{as } l \to +\infty$$

in the C^∞-sense, with some smooth coefficient functions $a_j(z)$, and $a_0(z) = 1$.
Asymptotics of the Bergman kernel: Berezin (for bounded symmetric domains (1974), Tian (1990) and Ruan (1998) (answering a conjecture of Yau) and Catlin (1999) and Zelditch (1998) (for compact complex manifolds), Englǐ(2002) (for a strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary and h subject to some technical hypotheses),

TYZ Expansion

In the context of our Kepler manifold, this has been studied by Gramchev and Loi (2009) for $h(z) = e^{-|z|}$ (so $\omega = \frac{i}{2}\partial\overline{\partial}|z|$; this turns out to be the symplectic form inherited from the isomorphism $\mathbb{H} \cong T^*\mathbb{S}^n \setminus \{\text{zero section}\}$ as abserved by Rawnsley 1977, who showed that

$$\epsilon_l(z) = l^n + \frac{(n-2)(n-1)}{2|z|} l^{n-1} + \sum_{k=2}^{n-2} \frac{b_k}{|z|^k} l^{n-k} + R_l(z), \qquad (3)$$

with some constants b_k independent of z and remainder term $R_l(z) = O(e^{-cl|z|})$ for some $c > 0$ (i.e. exponentially small).

TYZ Expansion

Theorem

Let $K_s = K_{\rho\otimes\mu}$ *for*

$$d\rho(t) = 2cme^{-st^{2m}}t^{2mn-1}\,dt, \tag{4}$$

where c, m *are fixed positive constants and* $s > 0$. *Then as* $s \to +\infty$,

$$e^{-s|z|^{2m}}K_s(z,z) = \frac{2m^n}{(n-1)!c}s^n\sum_{j=0}^{n-1}\frac{b_j}{s^j|z|^{2mj}} + R_s(z), \tag{5}$$

where b_j *are constants depending on* m *and* n *only,*

$$b_0 = 1, \quad b_1 = \frac{(1-n)(mn-n+1)}{2m},$$

and $R_s(z) = O(e^{-\delta s|z|^{2m}})$ *with some* $\delta > 0$.

TYZ Expansion

The result of Gramchev and Loi corresponds to $m = \frac{1}{2}$ and $c = \frac{2^{1-n}}{(n-1)!}$, so it is recovered as a special case. Our method of proof is a good deal simpler than and covers all $m > 0$, and is also extendible to other situations. We further note that $d(\rho \otimes \mu)$ with the ρ from (4) actually coincides (up to a constant factor) with ω^n for $\omega = \frac{i}{2}\partial\overline{\partial}(s|z|^{2m})$, in full accordance with (2) (taking $h(z) = e^{-|z|^{2m}}$, and with $l = 1, 2, \ldots$ replaced by the continous parameter $s > 0$ as already remarked above).

Hankel operators

The Hankel operator $H_{\overline{g}}$, $g \in A^2_{\rho\otimes\mu}$, is the operator from $A^2_{\rho\otimes\mu}$ into $L^2(\rho\otimes\mu)$ defined by

$$H_{\overline{g}}f := (I-P)(\overline{g}f),$$

where $P : L^2(\rho\otimes\mu) \to A^2_{\rho\otimes\mu}$ is the orthogonal projection. This is a densely defined operator, which is (extends to be) bounded e.g. whenever g is bounded.

Our third main result shows that for the Bergman space $A^2(\mathbb{M}) := A^2_{\rho\otimes\mu}$ for $d\rho(t) = \chi_{[0,1]}(t)t^{2n-1}dt$ on $\mathbb{M}$, there is also a cut-off at $p = 2n$.

Theorem

Let $p \geq 1$. Then the following are equivalent.

(i) *There exists nonconstant $g \in A^2(\mathbb{M})$ with $H_{\overline{g}} \in \mathcal{S}^p$.*

(ii) *There exists a nonzero homogeneous polynomial g of degree $m \geq 1$ such that $H_{\overline{g}} \in \mathcal{S}^p$.*

(iii) *There exists $m \geq 1$ such that $H_{\overline{g}} \in \mathcal{S}^p$ for all homogeneous polynomials g of degree m.*

(iv) $p > 2n$.

(v) $H_{\overline{g}} \in \mathcal{S}^p$ *for any polynomial g.*

Balanced metrics

h a potential on a bounded domain Ω in $\mathbb{C}^n$ which vanishes at the first order on the boundary:

$$h = 0 < \|\nabla h\| \quad \text{on} \quad \partial\Omega$$

and $\omega = \frac{i}{2}\partial\overline{\partial}\log\frac{1}{h}$ is a Kähler form on a Ω. The Hermitian metric associated to ω is called balanced if its Bergman kernel satisfies

$$h(z)K_{h\,\det[\partial\overline{\partial}\log\frac{1}{h}]}(z,z) \equiv \text{const.} \quad (\neq 0); \tag{6}$$

that is, if and only if the corresponding Kempf distortion function ϵ_1 is a nonzero constant. More generally, for $\alpha > n$ one calls ω *α-balanced* if it is balanced and h is commensurable to $\text{dist}(\cdot, \partial\Omega)^\alpha$ at the boundary. (For $\alpha \leq n$, the corresponding Bergman space degenerates just to the zero function, thus $K_{h\,\det[\partial\overline{\partial}\log\frac{1}{h}]} \equiv 0$ and the left-hand side in (6) is constant zero.) This definition turns out to indeed depend only on $\omega = \frac{i}{2}\partial\overline{\partial}\log\frac{1}{h}$ and not on the particular choice of h for a given ω, and also can be extended from domains to manifolds with line bundles as before; see Donaldson, Arezzo and Loi and Englis for further details.

Balanced metrics

The simplest example of α-balanced metric is $h(z) = (1-|z|^2)^\alpha$, $\alpha > 1$, on the unit disc $\mathbb{D}$ in $\mathbb{C}$; one then gets $\det[\partial\overline{\partial}\log\frac{1}{h}] = \frac{\alpha}{(1-|z|^2)^2}$ and $K_{h\det[\partial\overline{\partial}\log\frac{1}{h}]}(z,z) = \frac{\alpha-1}{\pi\alpha}(1-|z|^2)^{-\alpha}$, so that (6) holds with the constant $\frac{\alpha-1}{\pi\alpha}$. Similarly for $h(z) = (1-|z|^2)^\alpha$, $\alpha > n$, on the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$, where the constant turns out to be $\frac{\Gamma(\alpha)}{\Gamma(\alpha-n)\alpha^n\pi^n}$. The only known examples of bounded domains with balanced metrics are invariant metrics on bounded homogeneous domains (in particular, on bounded symmetric domains). In the unbounded setting, every dilation of the Euclidean metric on $\mathbb{C}^n$ is balanced (with $h(z) = e^{-\alpha|z|^2}$, $\alpha > 0$, so that $A^2(\mathbb{C}^n, h\det[\partial\overline{\partial}\log\frac{1}{h}])$ is just the familiar Fock space). Balanced metrics are known to exist in abundance on compact manifolds Donaldson; the existence of balanced metrics e.g. on bounded strictly pseudoconvex domains with smooth boundary is still an open problem, and their uniqueness for a given α is an open problem even on the unit disc.
We conclude this paper by the following simple observation concerning balanced metrics on our Kepler manifold $\mathbb{H}$. Note that for $n \geq 3$, $\mathbb{H}$ is known to be simply connected.

Balanced metrics

Theorem

Let $n = \dim_{\mathbb{C}} \mathbb{H} \geq 3$ and $\alpha > n$. Then either there does not exist any α-balanced metric on $\mathbb{H}$, or it is not unique.

Ideas of proofs

Theorem

If

$$\alpha_\phi(z) := \phi(|z|^2)\frac{\alpha(z)\wedge\overline{\alpha(z)}}{(-1)^{n(n+1)/2}(2i)^n}$$

then the corresponding weighted Bergman K_ϕ is given by

$$K_\phi(z,w) = \frac{1}{(n-1)!c_{\mathbb{M}}}\left[2tF^{(n-1)}(t) + (n-1)F^{(n-2)}(t)\right]_{t=z\bullet\overline{w}},$$

where

$$c_{\mathbb{M}} = (n-1)\int_{\mathbb{M}}\frac{\alpha(z)\wedge\overline{\alpha(z)}}{(-1)^{n(n+1)/2}(2i)^n}$$

and

$$F(t) = \sum_{k=0}^{\infty}\frac{t^k}{c_k}, \qquad \text{where } c_k := \int_0^{\infty} t^k\phi(t)\,dt. \tag{7}$$

it then follows that Mengotti-Y.

$$\int_{\partial\mathbb{M}} (z \bullet w)^k (\xi \bullet \overline{w})^l \, d\mu(w) = \delta_{kl} \frac{(z \bullet \xi)^k}{N(k)} \tag{8}$$

for all $z \in \mathbb{H}$, $\xi \in \mathbb{C}^{n+1}$; and, consequently, $\mathcal{P}_k$ and $\mathcal{P}_l$ are orthogonal in $L^2(d\mu)$ if $k \neq l$, while

$$\int_{\partial\mathbb{M}} f(w)(z \bullet \overline{w})^k \, d\mu(w) = \frac{f(z)}{N(k)} \tag{9}$$

for all $z \in \mathbb{H}$ and $f \in \mathcal{P}_k$.
If f is a function holomorphic on $R\mathbb{M}$ (for some $0 < R \leq \infty$), then it has a unique decomposition of the form

$$f = \sum_{k=0}^{\infty} f_k, \qquad f_k \in \mathcal{P}_k, \tag{10}$$