Some new fixed point results of monotone mappings and their applications

M. A. Khamsi

University of Texas at El Paso, USA.

King Fahd University of Petroleum & Minerals, Saudi Arabia.

Colloque de Mathématiques
En l' honneur des Mathématiciens Marocains à l' étranger
Faculté des Sciences - Kénitra, Morocco
22-23-24 Septembre 2016

Very often scientific branches which were thought to be completely disparate are suddenly seen to be related. This is the case for example with mathematics of which the level of sophistication applied to various sciences has changed drastically in recent years. Fixed point theory furnishes good example of a central concept with multitudes of different uses. It has always been a major theoretical tool in fields as widely apart as differential equations, topology, economics, game theory, dynamical systems (and chaos), optimal control, functional analysis, logic programming and artificial intelligence. Moreover, more or less recently, the usefulness of the concept for applications increased enormously by the development of accurate and efficient techniques for computing fixed points, making fixed point methods a major weapon in the arsenal of the applied mathematician.

The fixed point problem (at the basis of the Fixed Point Theory) may be stated as:

Problem

Let X be a set, A and B two nonempty subsets of X such that $A \cap B \neq \emptyset$, and $T : A \to B$ be a map. When does a point $x \in A$ exist such that $T(x) = x$?

A point x is called a fixed point of T whenever $T(x) = x$. The set of fixed points of T will be denoted $Fix(T)$.

Introduction to Fixed Point Theory

The fixed point problem (at the basis of the Fixed Point Theory) may be stated as:

Problem

Let X be a set, A and B two nonempty subsets of X such that $A \cap B \neq \emptyset$, and $T : A \to B$ be a map. When does a point $x \in A$ exist such that $T(x) = x$?

A point x is called a fixed point of T whenever $T(x) = x$. The set of fixed points of T will be denoted $Fix(T)$.

Historically, in connection with the fixed point problem, the following questions were investigated:

(1) Do we have the uniqueness of the fixed point?

(2) When a fixed point exists, how do we approximate it?

(3) If $Fix(T)$ is not empty, what is its structure?

Introduction to Fixed Point Theory

Fixed Point Theory is divided into three major areas:

1. Topological Fixed Point Theory
2. Metric Fixed Point Theory
3. Discrete Fixed Point Theory

[1] J.L. Berggren, *Episodes in the Mathematics of Medieval Islam*, Springer-Verlag, New York (1986).

Introduction to Fixed Point Theory

Fixed Point Theory is divided into three major areas:

1. Topological Fixed Point Theory
2. Metric Fixed Point Theory
3. Discrete Fixed Point Theory

Historically the boundary lines between the three areas was defined by the discovery of three major theorems:

1. Brouwer's Fixed Point Theorem (1910)
2. Banach's Fixed Point Theorem (1922 as early as 1429)[1]
3. Knaster-Tarski's Fixed Point Theorem (1955)

[1] J.L. Berggren, *Episodes in the Mathematics of Medieval Islam*, Springer-Verlag, New York (1986).

Banach Contraction Principle

In 1922 Banach published his fixed point theorem also known as **Banach's Contraction Principle** which uses the concept of Lipschitz mappings.

Definition

Let (M, d) be a metric space. The map $T : M \to M$ is said to be **Lipschitzian** if there exists a constant $k > 0$ (called Lipschitz constant) such that

$$d\Big(T(x), T(y)\Big) \leq k\ d(x, y)$$

for all $x, y \in M$. A Lipschitzian mapping with a Lipschitz constant k less than 1, i.e. $k < 1$, is called **contraction**, and **nonexpansive** when $k = 1$.

Banach's Contraction Principle

The theorem of Banach is the simplest and one of the most versatile results in fixed point theory. Being based on an iteration process, it can produce approximations of any required accuracy and, moreover, even the number of iterations needed to get specified accuracy can be determined.

Banach's Contraction Principle

The theorem of Banach is the simplest and one of the most versatile results in fixed point theory. Being based on an iteration process, it can produce approximations of any required accuracy and, moreover, even the number of iterations needed to get specified accuracy can be determined.

Theorem

Let (M, d) be a complete metric space and let $T : M \to M$ be a contraction mapping. Then T has a unique fixed point ω, and for each $x \in M$, we have

$$\lim_{n\to\infty} T^n(x) = \omega$$

Moreover,for each $x \in M$, we have

$$d\Big(T^n(x), \omega\Big) \leq \frac{k^n}{1-k}\, d\Big(T(x), x\Big)\ .$$

Quickly mathematicians wondered what happened to the conclusion of the Banach contraction principle when the Lipschitz constant is 1?

Quickly mathematicians wondered what happened to the conclusion of the Banach contraction principle when the Lipschitz constant is 1?

Example

(1) A rotation on the unit circle $\mathbb{U}$ in $\mathbb{R}^2$ may fail to have a fixed point though it is an isometry. In particular if π is the angle of the rotation, then we have $T(x) = -x$ and $\|x - T(x)\| = 2$, for any $x \in \mathbb{U}$.

Quickly mathematicians wondered what happened to the conclusion of the Banach contraction principle when the Lipschitz constant is 1?

Example

(1) A rotation on the unit circle $\mathbb{U}$ in $\mathbb{R}^2$ may fail to have a fixed point though it is an isometry. In particular if π is the angle of the rotation, then we have $T(x) = -x$ and $\|x - T(x)\| = 2$, for any $x \in \mathbb{U}$.

Clearly the fixed point is missed because $\mathbb{U}$ is not convex.

Example

(2) Let $X = \mathcal{C}([0,1])$ be the space of continuous functions defined on $[0,1]$ endowed with the sup-norm. Consider the set

$$C = \{f \in X;\ f(1) = 1 \textit{ and } 0 \leq f(x) \leq 1,\ \textit{for all } x \in [0,1]\}.$$

Then C is a nonempty closed convex subset of X which is bounded. Define the map $T : C \to C$ by

$$T(f)(x) = x\ f(x),\ \ x \in [0,1].$$

Then T is nonexpansive with an empty fixed point set.

Example

(2) Let $X = \mathcal{C}([0,1])$ be the space of continuous functions defined on $[0,1]$ endowed with the sup-norm. Consider the set

$$C = \{f \in X;\ f(1) = 1 \ \textit{and}\ 0 \leq f(x) \leq 1,\ \textit{for all}\ x \in [0,1]\}.$$

Then C is a nonempty closed convex subset of X which is bounded. Define the map $T : C \to C$ by

$$T(f)(x) = x\ f(x),\ \ x \in [0,1].$$

Then T is nonexpansive with an empty fixed point set. Indeed, if $f \in Fix(T)$, then we must have $f(x) = x\ f(x)$, for all $x \in [0,1]$. In particular, we have $f(x) = 0$ for $x \in [0,1)$. Since $f(x)$ is continuous, then we must have $f(1) = 0$. This is contradictory to the assumption $f \in C$.

A combination of the Brouwer and Banach fixed point theorems led to the so-called metric fixed point problem in Banach spaces:

A combination of the Brouwer and Banach fixed point theorems led to the so-called metric fixed point problem in Banach spaces:

Let X be a Banach space, and C a nonempty bounded closed convex subset of X. When does any nonexpansive mapping $T : C \to C$ have a fixed point?

The origin of metric fixed point theory for nonexpansive mappings as a noteworthy avenue of research almost surely dates from the 1965 publication of three fixed point theorems first of which are the two fixed point results discovered independently by Browder[1] and Göhde [2]:

[1] F. E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. U.S.A., **54** (1965), 1041-1044.

[2] D. Göhde, *Zum Prinzip der kontraktiven Abbildung,* Math. Nachr. **30** (1965), 251-258.

The origin of metric fixed point theory for nonexpansive mappings as a noteworthy avenue of research almost surely dates from the 1965 publication of three fixed point theorems first of which are the two fixed point results discovered independently by Browder[1] and Göhde [2]:

Theorem

If K is a bounded closed convex subset of a uniformly convex Banach space X and if $T : K \to K$ is nonexpansive, then T has a fixed point. Moreover the fixed point set of T is a closed convex subset of K.

---

[1] F. E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. U.S.A., **54** (1965), 1041-1044.

[2] D. Göhde, *Zum Prinzip der kontraktiven Abbildung,* Math. Nachr. **30** (1965), 251-258.

The third fixed point theorem was discovered by Kirk[1]:

Theorem

Let K be a weakly-compact convex subset of a Banach space X. Assume that K has the normal structure property. Then any nonexpansive mapping $T : K \to K$ has a fixed point.

[1] W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly 72(1965), 1004-1006.

The third fixed point theorem was discovered by Kirk[1]:

Theorem

Let K be a weakly-compact convex subset of a Banach space X. Assume that K has the normal structure property. Then any nonexpansive mapping $T : K \to K$ has a fixed point.

Definition

A closed convex subset C of a Banach space X is said to have the **normal structure property** if any bounded convex subset K of C which contains more than one point, contains a **nondiametral point**, i.e. there exists a point $x_0 \in K$ such that

$$\sup_{x \in K} \|x_0 - x\| < \mathrm{diam}(K)\ .$$

[1] W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly 72(1965), 1004-1006.

There is a fundamental difference between Browder and Göhde theorem with Kirk's theorem. The first two are constructive and use an algorithm while Kirk's theorem is set theoretical in nature.

There is a fundamental difference between Browder and Göhde theorem with Kirk's theorem. The first two are constructive and use an algorithm while Kirk's theorem is set theoretical in nature.
We know that a nonexpansive map defined on a closed convex bounded subset C of a Banach space X may have no fixed point. But approximate fixed points always exist. Indeed, let $T : C \to C$ be nonexpansive. Fix $x_0 \in C$ and $\lambda \in (0, 1)$. Define the mapping $T_\lambda : C \to C$ by

$$T_\lambda(x) = (1 - \lambda)x_0 + \lambda T(x),$$

for any $x \in C$. It is clear that T_λ is a contraction. Then 'T_λ has a unique fixed point $x_\lambda \in C$, i.e.

$$x_\lambda = (1 - \lambda)x_0 + \lambda T(x_\lambda).$$

Moreover, we have $x_\lambda - T(x_\lambda) = (1-\lambda)(x_0 - T(x_\lambda))$. Therefore, if C is bounded, i.e., $\delta(C) = diam(C) = \sup\{\|x-y\|; x, y \in C\} < +\infty$, then we have $\|x_\lambda - T(x_\lambda)\| \leq (1-\lambda)\, \delta(C)$, which implies

$$\inf\{\|x - T(x)\|; x \in C\} \leq (1-\lambda)\, \delta(C),$$

for any $\lambda \in (0,1)$. Hence $\inf\{\|x - T(x)\|; x \in C\} = 0$.

Moreover, we have $x_\lambda - T(x_\lambda) = (1-\lambda)(x_0 - T(x_\lambda))$. Therefore, if C is bounded, i.e., $\delta(C) = diam(C) = \sup\{\|x-y\|; x, y \in C\} < +\infty$, then we have $\|x_\lambda - T(x_\lambda)\| \leq (1-\lambda)\ \delta(C)$, which implies

$$\inf\{\|x - T(x)\|; x \in C\} \leq (1-\lambda)\ \delta(C),$$

for any $\lambda \in (0,1)$. Hence $\inf\{\|x - T(x)\|; x \in C\} = 0$.

Moreover, $\{T_\lambda^n(x)\}$ converges to x_λ, for any $x \in C$. Another representation of this fact will be useful later on. Indeed, fix $x_0 \in C$ and consider the sequence $\{x_n\}_{n\geq 1}$ in C defined by

$$x_{n+1} = (1-\lambda)x_0 + \lambda T(x_n), \quad n \geq 0.$$

Then $\{x_n\}$ converges to x_λ.

Ishikawa was the first to extend the previous iteration to obtain one of the most beautiful behavior of nonexpansive mappings.

Ishikawa was the first to extend the previous iteration to obtain one of the most beautiful behavior of nonexpansive mappings.

Lemma

Let $(X, \|.\|)$ be a Banach space, K a closed and convex nonempty subset of X, and $T : K \to K$ be nonexpansive. Let $\alpha \in (0,1)$. Fix $x_0 \in K$ and define the sequence $\{x_n\}$ by

$$x_{n+1} = (1-\alpha)x_n + \alpha\, T(x_n),\ \ n \in \mathbb{N}.$$

Set $y_n = T(x_n)$, $n \in \mathbb{N}$. Then for each $i, n \in \mathbb{N}$, we have

$$(1+n\alpha)\|y_i - x_i\| \leq \|y_{i+n} - x_i\| - (1-\alpha)^{-n}\Big[\|y_{i+n} - x_{i+n}\| - \|y_i - x_i\|\Big].$$

If K is bounded, then we have $\lim_{n\to+\infty} \|x_n - T(x_n)\| = 0$, i.e., $\{x_n\}$ is an approximate fixed point sequence of T.

Metric Fixed Point Theory in Banach Spaces

In their investigation, Browder and Göhde focused on reaching a fixed point via an approximate fixed point sequence. In particular, if X is a complete Hilbert space (like ℓ_2 for example), and if $\{x_n\}$ is any sequence in X which converges weakly (or coordinate-wise) to x, then we have

$$\liminf_{n\to\infty} \|x_n - y\|^2 = \liminf_{n\to\infty} \|x_n - x\|^2 + \|x - y\|^2,$$

for any $y \in X$. Using this wonderful property (a weaker version of which is known as Opial property [1]), Browder and Göhde proved that any weak-cluster point of an approximate fixed point sequence of a nonexpansive mapping T is a fixed point of T.

[1] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73:4** (1967),591-597.

Ran & Reurings Fixed Point Problem

Consider the matrix equation:

Problem

Find an $X \in H(n)$ so that

$$X = Q \pm \sum_{j=1}^{m} A_j^* F(X) A_j, \qquad (1)$$

where F is a **monotone function***, which maps $P(n)$ into itself.*

The study of equations like (1) is motivated by the fact that they often arise in the analysis of ladder networks, dynamic programming, control theory, stochastic filtering, statistics and many other applications[1].

[1] El-Sayed and Ran, *On an iteration method for solving a class of nonlinear matrix equations*, SIAM Journal on Matrix Analysis and Applications 23: 3 (2002), 632-645.

In their investigation of the equation (1), Ran and Reurings[1] came up with a new extension of the Banach Contraction Principle in partially ordered metric spaces.

[1] Ran and Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proceedings of the American Mathematical Society (2004), 1435-1443.

Ran & Reurings Fixed Point Theorem

Theorem

Let $(X, \preceq)$ be a partially ordered set such that every pair $x, y \in X$ has an upper and lower bound. Let d be a metric on X such that (X, d) is a complete metric space. Let $T : X \to X$ be a continuous monotone (either order preserving or order reversing) mapping. Suppose that the following conditions hold:

1. *There exists a $k \in (0, 1)$ with*
$$d(T(x), T(y)) \leq k\, d(x, y), \text{ for all } x \succeq y.$$
2. *There exists an $x_0 \in X$ with $x_0 \preceq T(x_0)$ or $x_0 \succeq T(x_0)$.*

Then T is a Picard Operator (PO), that is T has a unique fixed point $\omega \in X$ and for each $x \in X$, $\lim\limits_{n\to\infty} T^n(x) = \omega$.

Nieto-Rodríguez Fixed Point Problem

Nieto and Rodríguez extended Ran & Reurings theorem to find a periodic solution to the differential equation[1]:

$$\begin{cases} u'(t) = f(t, u(t)), & t \in [0, T] \\ u(0) = u(T), \end{cases}$$

where $T > 0$ and $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

[1] Nieto and Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order 22: 3

Nieto-Rodríguez Fixed Point Problem

Nieto and Rodríguez extended Ran & Reurings theorem to find a periodic solution to the differential equation[1]:

$$\begin{cases} u'(t) = f(t, u(t)), & t \in [0, T] \\ u(0) = u(T), \end{cases}$$

where $T > 0$ and $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. This equation is transformed to the integral equation:

$$u(t) = \int_0^T G(t, s)\Big(f(x, u(s)) + \lambda u(s)\Big) ds, \; t \in I,$$

with $\lambda > 0$, where the Green function $G(t, s)$ is given by

$$G(t, s) = \begin{cases} \dfrac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1}, & 0 \leq s < t \leq T, \\ \dfrac{e^{\lambda(s-t)}}{e^{\lambda T} - 1}, & 0 \leq t < s \leq T. \end{cases}$$

[1] Nieto and Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order 22: 3

But the fixed point obtained and used by Nieto et al. is useful despite this problem with his example.

Nieto-Rodríguez Fixed Point Problem

Therefore the solution to the differential equation is a fixed point of the operator $T : C(I, \mathbb{R}) \to C(I, \mathbb{R})$ defined by

$$T(u)(t) = \int_0^T G(t,s)\Big(f(x,u(s)) + \lambda u(s)\Big)ds,\ \ t \in I,$$

Nieto-Rodríguez Fixed Point Problem

Therefore the solution to the differential equation is a fixed point of the operator $T : C(I, \mathbb{R}) \to C(I, \mathbb{R})$ defined by

$$T(u)(t) = \int_0^T G(t,s)\Big(f(x, u(s)) + \lambda u(s)\Big) ds,\ t \in I,$$

Nieto et al. assumed that if the function f satisfies the condition

$$0 \leq f(t,u) + \lambda u - [f(t,v) + \lambda v] \leq \mu(u - v),$$

for some $0 < \mu < \lambda$ and for any $u, v \in \mathbb{R}$ such that $u \geq v$, then T has a unique fixed point.

Nieto-Rodríguez Fixed Point Problem

Therefore the solution to the differential equation is a fixed point of the operator $T : C(I, \mathbb{R}) \to C(I, \mathbb{R})$ defined by

$$T(u)(t) = \int_0^T G(t,s)\Big(f(x,u(s)) + \lambda u(s)\Big)ds, \; t \in I,$$

Nieto et al. assumed that if the function f satisfies the condition

$$0 \leq f(t,u) + \lambda u - [f(t,v) + \lambda v] \leq \mu(u - v),$$

for some $0 < \mu < \lambda$ and for any $u, v \in \mathbb{R}$ such that $u \geq v$, then T has a unique fixed point.
The only problem is that under these assumptions T is in fact a contraction on the entire space $C(I, \mathbb{R})$.

Nieto-Rodríguez Fixed Point Theorem

But the fixed point theorem obtained and used by Nieto et al. is useful despite this problem with his example.

Theorem

Let (X, d) be a complete metric space endowed with a partial order $\preceq$. Let $T : X \to X$ be an order preserving mapping such that there exists a $k \in [0, 1)$ with

$$d(T(x), T(y)) \leq k\, d(x, y), \text{ whenever } x \preceq y.$$

Assume that $(X, d, \preceq)$ satisfies the following property:

(*) *for any nondecreasing (nonincreasing) $(x_n)_{n\in\mathbb{N}}$, if $x_n \to x$, then $x_n \preceq x$ ($x \preceq x_n$), for $n \in \mathbb{N}$.*

If there exists an $x_0 \in X$ such that x_0 and $T(x_0)$ are comparable, then T has a fixed point. Moreover if $(X, \preceq)$ is such that every pair of elements of X has an upper or a lower bound, then T is a Picard Operator.

Jachymski Graphical Approach

Jachymski [1] was the first to recognize the power behind using graphs instead of partial orders. Let (X, d) be a metric space and let G be a directed graph such that $V(G) = X$ and $E(G) \supseteq \Delta$. Such digraphs are called reflexive. Let C be a nonempty subset of X. A mapping $T : C \to C$ is called

1. G-monotone if $(T(x), T(y)) \in E(G)$ whenever $(x, y) \in E(G)$;
2. G-contraction if T is G-monotone and there exists $k \in [0, 1)$ such that
$$d(T(x), T(y)) \leq k\, d(x, y), \quad \textit{whenever} \quad (x, y) \in E(G);$$
3. G-monotone nonexpansive if T is G-monotone and
$$d(T(x), T(y)) \leq d(x, y), \quad \textit{whenever} \quad (x, y) \in E(G).$$

[1] J. Jachymski, *The contraction principle for mappings on a metric space with a graph*, Proc. Amer. Math. Soc. 1 (136) (2008) 1359-1373.

The graphical formulation of Nieto et al. property (*) is:

(G*) for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, if $x_n \to x$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E(G)$, then $(x_n, x) \in E(G)$, for all $n \in \mathbb{N}$.

Theorem

Let (X, d) be a complete metric space. Assume that the triplet (X, d, G) has the property (G). Let $T : X \to X$ be a G-contraction. Denote by $X_T = \{x \in X; (x, T(x)) \in E(G)\}$. Then the following statements hold:*

(1) *$Fix(T) \neq \emptyset$ if and only if $X_T \neq \emptyset$;*

(2) *if $X_T \neq \emptyset$ and G is weakly connected, then T is a Picard Operator, i.e., $Fix(T) = \{x^*\}$ and $\{T^n(x)\} \to x^*$ as $n \to \infty$, for all $x \in X$.*

Monotone Nonexpansive Mappings

Let (X, d) be a metric space and let G be a reflexive directed graph such that $V(G) = X$. Let $T : X \to X$ be a G-monotone nonexpansive mapping. Assume there exists $x_0 \in X$ such that $(x_0, T(x_0)) \in E(G)$. Under what conditions T has a fixed point?

Monotone Nonexpansive Mappings

Let (X, d) be a metric space and let G be a reflexive directed graph such that $V(G) = X$. Let $T : X \to X$ be a G-monotone nonexpansive mapping. Assume there exists $x_0 \in X$ such that $(x_0, T(x_0)) \in E(G)$. Under what conditions T has a fixed point?

Similar to what happened in the general case, we expect some positive answers in Banach spaces.

Monotone Nonexpansive Mappings

Let (X, d) be a metric space and let G be a reflexive directed graph such that $V(G) = X$. Let $T : X \to X$ be a G-monotone nonexpansive mapping. Assume there exists $x_0 \in X$ such that $(x_0, T(x_0)) \in E(G)$. Under what conditions T has a fixed point?

Similar to what happened in the general case, we expect some positive answers in Banach spaces.

Since monotone mappings do not in general have good global behavior, it is hard to expect some general fixed point theorems, like Kirk's fixed point theorem, to extend easily. Therefore our approach was based on techniques that use constructive proofs.

Let $(X, \|.\|, G)$ be a Banach space endowed with a graph G such that G has some nice properties with respect to the linear structure of X. Let C be a bounded nonempty convex subset of X.

Let $(X, \|.\|, G)$ be a Banach space endowed with a graph G such that G has some nice properties with respect to the linear structure of X. Let C be a bounded nonempty convex subset of X. Let $T : C \to C$ be a G-monotone nonexpansive mapping. Fix $\lambda \in (0, 1)$. Let $x_0 \in C$ be such that $(x_0, T(x_0)) \in E(G)$. By induction we construct a sequence $\{x_n\}$ in C such that the following hold for any $n \geq 0$:

Let $(X, \|.\|, G)$ be a Banach space endowed with a graph G such that G has some nice properties with respect to the linear structure of X. Let C be a bounded nonempty convex subset of X. Let $T : C \to C$ be a G-monotone nonexpansive mapping. Fix $\lambda \in (0, 1)$. Let $x_0 \in C$ be such that $(x_0, T(x_0)) \in E(G)$. By induction we construct a sequence $\{x_n\}$ in C such that the following hold for any $n \geq 0$:

(i) $x_{n+1} = (1 - \lambda)x_n + \lambda\, T(x_n)$,

(ii) (x_n, x_{n+1}), $(x_{n+1}, T(x_n))$ and $(T(x_n), T(x_{n+1}))$ are in $E(G)$,

(iii) $\|T(x_{n+1}) - T(x_n)\| \leq \|x_{n+1} - x_n\|$.

Such sequence is known as Krasnoselskii sequence.

Let $(X, \|.\|, G)$ be a Banach space endowed with a graph G such that G has some nice properties with respect to the linear structure of X. Let C be a bounded nonempty convex subset of X. Let $T : C \to C$ be a G-monotone nonexpansive mapping. Fix $\lambda \in (0,1)$. Let $x_0 \in C$ be such that $(x_0, T(x_0)) \in E(G)$. By induction we construct a sequence $\{x_n\}$ in C such that the following hold for any $n \geq 0$:

(i) $x_{n+1} = (1-\lambda)x_n + \lambda\, T(x_n)$,

(ii) (x_n, x_{n+1}), $(x_{n+1}, T(x_n))$ and $(T(x_n), T(x_{n+1}))$ are in $E(G)$,

(iii) $\|T(x_{n+1}) - T(x_n)\| \leq \|x_{n+1} - x_n\|$.

Such sequence is known as Krasnoselskii sequence. The Mann iteration is an extension of Krasnoselskii iteration defined by:

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n\, T(x_n),$$

where x_0 is a starting point, $\lambda_n \in (0,1)$ and $n \geq 0$.

Monotone Nonexpansive Mappings in Banach spaces

Let X be a Banach space and τ a topology on X.

Definition

X is said to satisfy the τ-Opial condition if for any sequence $\{y_n\}$ in X which τ-converges to y, we have

$$\limsup_{n\to+\infty} \|y_n - y\| < \limsup_{n\to+\infty} \|y_n - z\|,$$

for any $z \in X$ such that $z \neq y$.

Let X be a Banach space and τ a topology on X.

Definition

X is said to satisfy the τ-Opial condition if for any sequence $\{y_n\}$ in X which τ-converges to y, we have

$$\limsup_{n\to+\infty} \|y_n - y\| < \limsup_{n\to+\infty} \|y_n - z\|,$$

for any $z \in X$ such that $z \neq y$.

Any Hilbert space and the classical ℓ_p spaces, $p > 1$, all satisfy the weak-Opial condition. Recall the definition of ℓ_p spaces:

$$\ell_p = \left\{ (x_n) \in \mathbb{R}^{\mathbb{N}},\ \sum_n |x_n|^p < +\infty \right\}.$$

Jachymski introduced a weaker version of the property (G*) trying to improve on the (*) property introduced by Nieto et al.

Definition

The triplet $(X, \|.\|, G)$ has property (P_τ) if and only if for any sequence $\{x_n\}_{n \in \mathbb{N}}$ in X such that $(x_n, x_{n+1}) \in E(G)$, for any $n \in \mathbb{N}$, and if a subsequence $\{x_{k_n}\}$ τ-converges to x, then $(x_{k_n}, x) \in E(G)$, for all $n \in \mathbb{N}$.

Jachymski introduced a weaker version of the property (G*) trying to improve on the (*) property introduced by Nieto et al.

Definition

The triplet $(X, \|.\|, G)$ has property (P_τ) if and only if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that $(x_n, x_{n+1}) \in E(G)$, for any $n \in \mathbb{N}$, and if a subsequence $\{x_{k_n}\}$ τ-converges to x, then $(x_{k_n}, x) \in E(G)$, for all $n \in \mathbb{N}$.

Earlier, we said that the graph G has some nice behavior vis-á-vis the linear structure of X. This behavior involves the G-intervals.

Definition

A G-interval is any of the subsets $[a, \rightarrow) = \{x \in C; (a, x) \in E(G)\}$ and $(\leftarrow, b] = \{x \in C; (x, b) \in E(G)\}$, for any $a, b \in C$.

Our first result in this direction is an analogue to Opial's fixed point theorem and its extensions.

Theorem

Let X be a Banach space which satisfies the τ-Opial condition. Assume that $(X, \|.\|, G)$ has property (P_τ) and the G-intervals are convex. Let C be a bounded convex τ-compact nonempty subset of X not reduced to one point. Let $T : C \to C$ be a G-monotone nonexpansive mapping. Assume there exists $x_0 \in C$ such that $(x_0, T(x_0)) \in E(G)$. Then T has a fixed point.

The Banach space ℓ_p, $p > 1$ satisfies the weak-Opial condition. Consider the digraph G on ℓ_p defined by

$$(\{\alpha_n\}, \{\beta_n\}) \in E(G) \Longleftrightarrow \alpha_n \leq \beta_n, n \geq 1.$$

As a Corollary to the previous theorem, we get the following result:

The Banach space ℓ_p, $p > 1$ satisfies the weak-Opial condition. Consider the digraph G on ℓ_p defined by

$$(\{\alpha_n\}, \{\beta_n\}) \in E(G) \Longleftrightarrow \alpha_n \leq \beta_n, n \geq 1.$$

As a Corollary to the previous theorem, we get the following result:

Corollary

Let C be a bounded closed convex nonempty subset of ℓ_p, $1 < p < +\infty$. Then any G-monotone nonexpansive mapping $T : C \to C$ has a fixed point provided there exists a point $x_0 \in C$ such that $(x_0, T(x_0)) \in E(G)$ or $(T(x_0), x_0) \in E(G)$.

It is known that L_p, $p \geq 1$ and $p \neq 2$, fails the weak-Opial condition. But if τ is the almost everywhere convergence topology, then L_p satisfies the τ-Opial condition. Consider the digraph G on L_p defined by

$$(f, g) \in E(G) \Longleftrightarrow f(t) \leq g(t) \textit{ almost everywhere}.$$

Again as a Corollary to the previous theorem, we get the following result:

It is known that L_p, $p \geq 1$ and $p \neq 2$, fails the weak-Opial condition. But if τ is the almost everywhere convergence topology, then L_p satisfies the τ-Opial condition. Consider the digraph G on L_p defined by

$$(f, g) \in E(G) \iff f(t) \leq g(t) \text{ almost everywhere.}$$

Again as a Corollary to the previous theorem, we get the following result:

Corollary

Let C be a bounded closed convex nonempty subset of L_p, $1 \leq p < +\infty$. Assume C is almost everywhere compact. Then any G-monotone nonexpansive mapping $T : C \to C$ has a fixed point provided there exists a point $f_0 \in C$ such that $(f_0, T(f_0)) \in E(G)$ or $(T(f_0), f_0) \in E(G)$.

Browder and Göhde fixed point theorem for monotone nonexpansive mappings

It is well known that L_p, $p > 1$, are uniformly convex. Therefore a similar result to Browder and Göhde fixed point theorem for monotone nonexpansive mappings will be very useful.

Browder and Göhde fixed point theorem for monotone nonexpansive mappings

It is well known that L_p, $p > 1$, are uniformly convex. Therefore a similar result to Browder and Göhde fixed point theorem for monotone nonexpansive mappings will be very useful.
Very recently, we were able to prove the following result:

Theorem

Let $(X, \|.\|)$ be a Banach space which is uniformly convex. Let G be a directed reflexive and transitive digraph defined on X such that G-intervals are closed and convex. Let C be a nonempty weakly compact convex subset of X and $T : C \to C$ be a G-monotone nonexpansive mapping. Assume there exists $x_0 \in C$ such that $(x_0, T(x_0)) \in E(\widetilde{G})$. Then T has a fixed point.

Browder and Göhde fixed point theorem for monotone nonexpansive mappings

It is well known that L_p, $p > 1$, are uniformly convex. Therefore a similar result to Browder and Göhde fixed point theorem for monotone nonexpansive mappings will be very useful.
Very recently, we were able to prove the following result:

Theorem

Let $(X, \|.\|)$ be a Banach space which is uniformly convex. Let G be a directed reflexive and transitive digraph defined on X such that G-intervals are closed and convex. Let C be a nonempty weakly compact convex subset of X and $T : C \to C$ be a G-monotone nonexpansive mapping. Assume there exists $x_0 \in C$ such that $(x_0, T(x_0)) \in E(\widetilde{G})$. Then T has a fixed point.

Most of the above results hold in hyperbolic metric spaces like $CAT(0)$ spaces.

Open Problems

1. Do we have a result similar to Kirk's fixed point theorem for monotone nonexpansive mappings?
2. Do we a have general structure for infinite digraphs which allows for the existence of a fixed point without involving the geometry of the underlying space?

Open Problems

1. Do we have a result similar to Kirk's fixed point theorem for monotone nonexpansive mappings?
2. Do we a have general structure for infinite digraphs which allows for the existence of a fixed point without involving the geometry of the underlying space?

Any questions?