Canonical metrics on projective varieties

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What is a canonical metric?

Let X be a compact Kähler manifold of dimension $n \geq 1$. A fundamental problem in Kähler Geometry is to find a Kähler metric g on X with "nice properties" that reflect the topological and the geometric properties of the manifold X, e.g.

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- existence of a constant scalar curvature Kähler metric ($cscK$): this is an open problem $\Longleftrightarrow$ a non linear PDE of 4^{th} order on the potential,
- existence of a Kähler-Einstein metric i.e. a metric whose Ricci curvature tensor is proportional to the metric tensor.

The goal of this lecture is to explain the Calabi conjecture as well as the problem of the existence of Kähler-Einstein metrics on compact Kähler manifolds and to show how this problems boil down to solving complex Monge-Ampère equations. We state the important theorems by Calabi-Yau ([Yau78]) and Aubin-Yau [Au78]) who gave the solutions to these problems in the case of zero first Chern class and negative first Chern class respectively.
Then we show how to extend these results to the case of normal projecive varieties with mild singularities ([EGZ09]). The consideration of singular varieties appears naturally in the classification of complex algebraic manifolds of high dimension (MMP), where the canonical or minimal models are singular with mild singularities (see [BCHM10]).
To undestand what can be expected, let us look at the simplest case of dimension $n = 1$ i.e. X is a compact Riemann surface.

Let h be a hermitian metric on X. In a local coordinate chart (U, z), we have:

$$h_{|U} = H\, dz \otimes d\bar{z},$$

where $H > 0$ is a smooth positive function in the open set U.
Recall that the Gaussian curvature of the Riemann surface (X, h) is a global smooth function on X, given locally in a chart (U, z) by the formula

$$K_{h|U} := -\frac{\Delta log H}{2H},$$

where $\Delta := 4\frac{\partial^2}{\partial z \partial \bar{z}}$ is the euclidean Laplace operator in the coordinate z.
By the Gauss-Bonnet formula, we have

$$\frac{1}{2\pi} \int_X K_h \omega = \chi(X) = 2 - 2g(X),$$

where $\chi(X)$ is the Euler-Poincaré characteristic of X and $g(X) \in \mathbb{N}$ is the genus of the compact surface X.

The existence of a metric h with constant Gaussian curvature $K_h \equiv \lambda \in \mathbb{R}$ implies that $\chi(X) = \lambda \frac{1}{2\pi} \int_X \omega$ has the same sign as the constant λ.

By the Uniformization Theorem any compact Riemann surface admits a Gaussian constant curvature metric. More precisely it gives a trichotomy in terms of the genus of the surface X.

- If $\chi(X) > 0$ i.e. $g(X) = 0$ then $X \simeq \mathbb{P}^1(\mathbb{C}) \simeq \mathbb{S}^2$ is the Riemann sphere, hence X carries a metric with positive constant Gauss curvature (the round metric),

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- If $\chi(X) = 0$ i.e. $g(X) = 1$ then $X \simeq \mathbb{C}/\Lambda$ is an elliptic curve (a complex torus), hence X carries a flat metric i.e. with Gaussian constant curvature 0.
- If $\chi(X) < 0$ i.e. $g(X) \geq 2$, then $X \simeq \mathbb{H}/\Gamma$ is a hyperbolic surface (Γ being Fuschian group of Möbius transformations of the Poincaré half-plane), hence it carries a hyperbolic metric of constant negative Gaussian curvature.

Recall that the genus $g(X)$ is a topological invariant of the surface and by Hodge theory it can be expressed as the dimension of the space of holomorphic forms on X:

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This number measures the growth of the dimension of the spaces of holomorphic pluricanonical sections i.e. holomorphic sections of the positive powers of the canonical bundle

$$P_m(X) := \dim H^0(X, K_X^m) \sim m^\kappa, \text{ as } m \to +\infty, \; K_X = \bigwedge^n (T^{1,0}X)^*.$$

It's know that $\kappa(X) \in \{-\infty, 0, 1, \cdots, n\}$ is a birational invariant and has the following rough meaning:

- The case of negative Kodaira dimension ($\kappa(X) = -\infty$) corresponds to positive curvature (in some direction, not necessarily all directions); this is the case for uniruled manifolds i.e. manifolds covered by rational curves (e.g. $\mathbb{P}^1 \times Y$ for any compact Kähler manifold Y),

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- the case $\kappa(X) = n$ corresponds to "negative curvature" i.e. the canonical bundle K_X is big (e.g. $c_1(X) = -c_1(K_X) < 0$): X is said to be of *general type*. This generalizes the hyperbolic Riemann surfaces ($g(X) \geq 2$ i.e. $\chi(X) < 0$).

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- the case of intermediate Kodaira dimension between $1 \leq \kappa(X) \leq n-1$ is not well understood: it should corresponds to negatively curved in some directions, flat in others (e.g. $\mathcal{E} \times Y$, where $\mathcal{E}$ is an elliptic curve and Y is a hyperbolic surface).

The Calabi conjecture and the Kähler-Einstein problem

In conclusion, very roughly speaking the Kodaira dimension called also the *canonical dimension* gives the number of "directions of negative curvature".

In dimension $n = 1$, we have $\kappa(X) = -\infty$ iff $X \simeq \mathbb{P}^1$, $\kappa(X) = 0$ iff X is an elliptic curve and $\kappa(X) = 1$ iff X is a hyperbolic surface.

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so that its fundamental form $\omega = \omega_h$ is given locally by

$$\omega_{|U} = \sum_{j,k} h_{j\bar{k}} \sqrt{-1} dz_j \wedge d\bar{z}_k.$$

The metric h is said to be a Kähler metric if its fundamental form is closed i.e. $d\omega = 0$. This condition means that locally in a neighbourhood of each point $z_0 \in X$ we can find normal complex coordinates i.e. h is tangent to the identity tensor at the point z_0 up to the second order.

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The associated volume form of the metric is given locally by the formula:

$$dV_\omega = \omega^n / n! = 2^n \det \left(h_{j\bar{k}} \right) dV_e(z),$$

where $dV_e(z)$ is the eulidean volume form in the local coordinates $(z_1, \cdot, z_n)$.

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Recall that the complex structure on X induces a natural decomposion of the usual exterior diffential operator $d = \partial + \bar{\partial}$ into complex conjugate operators. In local coordinates, we have,

$$\partial = \sum_{j=1}^{n} \frac{\partial}{\partial z_j} dz_j, \qquad \bar{\partial} = \sum_{j=1}^{n} \frac{\partial}{\partial \bar{z}_j} d\bar{z}_j.$$

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$$\mathrm{Ric}\,(\omega)_{|U} := -dd^c \log\det\left(h_{j\bar k}\right) = -\frac{1}{\pi}\sum_{j,k}\frac{\partial^2}{\partial z_j \partial \bar z_k}\log\det\left(h_{j\bar k}\right)\sqrt{-1}dz_j \wedge d\bar z_k.$$

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It is easy to see that this form is invariant under holomorphic change of coordinates and then it defines a global smooth d-closed $(1,1)$-form on X. Writing for convenience locally $\mathrm{Ric}\,\omega = -dd^c \log \omega^n$, we can easily check that if ω and $\tilde\omega$ are two Kähler metrics on X then

$$\mathrm{Ric}\ \omega - \mathrm{Ric}\ \tilde\omega = dd^c \log(\tilde\omega^n/\omega^n),$$

where $(\tilde\omega^n/\omega^n)$ is a smooth positve global function on X.

Therefore the de Rham cohomology class $\{\mathrm{Ric}\,\omega\}$ in $H^{1,1}(X,\mathbb{R}) \subset H^2(X,\mathbb{R})$ is independent of the Kähler metric ω.
It turns out to that this cohomology class is equal to the first Chern class of X i.e.

$$c_1(X) = \{\mathrm{Ric}\,\omega\} = -c_1(K_X), \;\; K_X = \bigwedge^n (T^{1,0}X)^*.$$

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We can now state the Calabi conjecture ([Cal57]).

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This conjecture has been solved by S.T. Yau in 1976 (see[Y76]). Actually E. Calabi adressed other related problems which played a fundamental role in the developement of Kähler geometry.

To state the Kähler-Einstein problem, we need to introduce some definitions.
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$$\mathrm{Ric}\ \omega = (K_\omega/2\pi)\ \omega,$$

thus a KE metric on X is a metric of constant Gaussian curvature and by the Gauss-Bonnet formula we have

$$\chi(X) = \int_X \mathrm{Ric}\,\omega = \int_X c_1(X).$$

As in dimension 1, the existence of of a KE metric on X imposes very strong conditions on the topology of X. Namely if ω is a Kähler-Einstein metric then $c_1(X) = \{\mathrm{Ric}\ \omega\} = \{\lambda\omega]$ contains $\lambda\omega$ i.e. $c_1(X)$ has a fixed sign given by the sign of λ.

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- When $\lambda = -1$ and there exists a Kähler metric ω such that $-\omega \in c_1(X)$, we say $c_1(X) < 0$ i.e. the canonical bundle is positive $K_X > 0$. In this case $\kappa(X) = n$ is maximal and X is is of general type.

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- When $\lambda = 0$ i.e. $c_1(X) = 0 = -c_1(K_X)$, i.e. the canonical bundle is trivial $K_X \equiv 0$, X is a Calabi-Yau manifold. In this case $\kappa(X) = 0$.

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- When $\lambda = +1$ and $\omega \in c_1(X)$, we say that $c_1(X) > 0$ and X is said to be a Fano manifold. In this case $\kappa(X) = -\infty$.

The Kähler-Einstein problem: *Assume that* X *is a Kähler manifold such that its first Chern class has a sign (i.e.* $c_1(X) = 0$*, or* $c_1(X) < 0$ *or* $c_1(X) > 0$*). Does* X *admit a Kähler-Einstein metric?*

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S.T. Yau was able to solve this problem in 1976 by proving the existence of a unique Ricci-flat Kähler metric in each Kähler class when $c_1(X) = 0$.

When $c_1(X) < 0$, Aubin and Yau independently proved the existence of a unique KE metric on X.

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When $c_1(X) > 0$ (X is said to be a Fano manifold) , there are obstructions to the existence of a KE metric and there is not uniqueness by invariance of the KE equation by automorphisms of X. Yau observed that the complex projective plane blown up at 2 points has no Kähler-Einstein metric and formulated a general conjecture.

Reduction to a complex Monge-Ampère equation

We will need the following important lemma called the dd^c-lemma which is consequence of Hodge theory.

Lemma

A d-closed smooth real $(1,1)$-form on X is d-exact iff it is dd^c-exact i.e. the cohomology class of any real $(1,1)$-form α on X is given by

$$\{\alpha\} = \{\alpha + dd^c\varphi; \varphi \in C^\infty(X,\mathbb{R})\} \simeq C^\infty(X,\mathbb{R})/\mathbb{R}.$$

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Namely let $\eta \in c_1(X)$ be a given closed smooth real $(1,1)$-form on X and fix a Kähler metric ω_0 on X.

The idea is to deform the metric ω_0 into a new metric within its cohomology class $\{\omega_0\}$ until we hopefully reach a metric ω such that $\mathrm{Ric}\,\omega = \eta$.

Indeed by the dd^c-lemma, since ω varies in the cohomology class of ω_0, there exists a smooth function $\varphi \in C^\infty(X, \mathbb{R})$ (unique up to an additive constant) such that $\omega = \omega_0 + dd^c\varphi =: \omega_\varphi$.

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On the other hand, since $\mathrm{Ric}\,\omega_0 \in c_1(X)$ and $\eta \in c_1(X)$ it follows that there exists a smooth function $\rho \in C^\infty(X, \mathbb{R})$ (unique up to an additive constant) such that $\mathrm{Ric}\,\omega_0 - \eta = dd^c\rho$.

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Hence

$$Ric\, \omega_\varphi = \eta \Longleftrightarrow dd^c \log(\omega_\varphi^n/\omega_0^n) = dd^c\rho \Longleftrightarrow dd^c[\log(\omega_\varphi^n/\omega_0^n - \rho] = 0.$$

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This is fully non linear second order PDE on X. It is elliptic non degenerate if we restrict to the open convex set of Kähler potentials defined by

$$\mathcal{P}^+(X, \omega_0) := \{\varphi \in C^\infty(X, \mathbb{R}); \omega_\varphi := \omega_0 + dd^c\varphi > 0\}.$$

Theorem

(Calabi-Yau Theorem, 1976). Let $0 < f \in C^\infty(X, \mathbb{R})$ *such that* $\int_X f \omega_0^n = \int_X \omega_0^n$*. Then there exists a unique function* $\varphi \in C^\infty(X, \mathbb{R})$ *such that* $\omega_0 + dd^c\varphi > 0$ *satisfying the equation* (MA) *and normalized by* $\int_X \varphi \omega_0^n = 0$.

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The uniqueness was proved by Calabi who suggested the use of the continuity method to prove the existence. It consists in connecting the equation to be solved to another equation that we know how to solve. Namely, consider for each parameter $t \in [0, 1]$, the following Monge-Ampère equation

$$(MA)_t \qquad (\omega_0 + dd^c\varphi_t)^n = (1 - t)\omega_0 + tf\omega_0^n.$$

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Now the goal is to consider the set S of papameters $t \in [0, 1]$ such that the equation $(MA)_t$ has a unique solution satisfying $\int_X \varphi_t\,\omega_0^n = 0$ and prove that $1 \in S$.

Observe that $0 \in S$ since the function $\varphi_0 = 0$ is a trivial solution to $(MA)_0$. Therefore all remains to show that S is open and closed in $[0, 1]$.

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The more delicate issue is to prove the closeness. It is done using Ascoli-Arzela theorem and rely on establishing uniform a priori estimates of any order on the solutions. The most delicate one being the C^0-uniform a priori estimate (see [Y76]).

In the same way the KE equation can be reduced to a complex Monge-Ampère equation of the following type:

$$(CMA)_\lambda, \qquad \omega_\varphi^n = e^{-\lambda\varphi} e^\rho \omega_0^n,$$

where ρ is a smooth function on X such that $\mathrm{Ric}\,\omega_0 - \lambda\omega_0 = dd^c\rho$ normalized by the condition $\int_X e^\rho \omega_0^n = 0$.

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Observe that when $c_1(X) = 0$, we can take $\lambda = 0$ and then this equation reduces to the equation (MA). Hence Calabi-Yau's theorem implies that in any Kähler class there is a unique Ricci flat Kähler metric ω_φ.

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Theorem

(Aubin-Yau Theorem 1978). Let ω_0 be any Kähler metric on X and $0 < f \in C^\infty(X, \mathbb{R})$ such that $\int_X f\omega_0^n = \int_X \omega_0^n$. Then there exists a unique $\varphi \in C^\infty(X, \mathbb{R})$ such that $\omega_0 + dd^c\varphi > 0$ and satisfying the following equation:

$$(\omega_0 + dd^c\varphi)^n = e^\varphi f \omega_0^n.$$

Singular versions of the CY and AY theorems

Let V be a normal projective variety with "mild singularities", those precisely which show up in the Minimal Model Program (classification of projective manifolds up to birational isomorphism) .

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Recall that a singular space is locally embedded as an analytic subset of some C^N. Therefore we can define smooth, holomorphic, meromorhic functions locally as restrictions of the corresponding functions in a local embedding. Therefore the notion of currents make sense on V.

If the space is normal, local holomorphic (meromorphic) functions on V^{reg} extend into local holomorphic (resp. meromorphic) functions on V. But the pull back of these objects to any (smooth) resolution of singualrities of V gets zeros and poles along the exceptional divisor.

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- Example : $\sum_{j=0}^n z_j^2 = 0 \longleftrightarrow$ the ordinary double point.
- This is not a quotient singularity if $n \geq 3$.

In this situation the canonical bundle K_V can be defined on V^{reg} as a holomorphic line bundle or as a divisor (up to linear equivalence) or as a sheaf of germs of local holomorphic n-forms on V.

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We can also define the Ricci curvature of a Kähler metric on V as a closed positive $(1,1)$-current on V with bounded local potentiels which is a smooth real $(1,1)$-form on the complex manifold V^{reg} of regular points of V. Observe that the local potentials of this current may have discontinuities on the singular part V^{sing} of V.

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The Calabi conjecture as well as the problem of existence of a (singular) KE metric on V can theen be formulated in the same way in this context.

Theorem

([EGZ09]). Let V be a normal projective variety with klt singularities.
1. Assume that K_V is an ample divisor. Then there exists a unique singular KE metric S_{KE} on V i.e. $Ric\, S_{KE} = -S_{KE}$ on V^{reg}.
2. If $K_V = 0$, then in each Kähler class on V, there exists a unique Ricci-flat singular Kähler metric S_{KE} on V i.e. $Ric\, S_{KE} = 0$ in V^{reg}.
Moreover in each case the potential of the singular metric is a locally bounded quasi-plurisubharmonic function on V which is a smooth KE metric on the complex manifold V^{reg}.

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The first step of the proof consists as before in reducing the KE equation $\mathrm{Ric}\,\omega = \lambda\omega$ to a complex Monge-Ampère equation:

$$(\omega_V + dd^c\varphi)^n = e^{\varepsilon\varphi} e^{\rho} \omega_V^n,$$

where ω_V is a Kähler metric on V and $\varepsilon = -\lambda \geq 0$.

To solve the above complex Monge-Ampère equation, we pull it back to a smooth log-resolution $\pi : X \longrightarrow V$.

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Letting $\theta := \pi^*(\omega_V)$, we see that this equation can be written as

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where

- $u := \varphi \circ \pi$ is the unknown function which is a bounded θ-plurisubharmonic function i.e. u is quasi-psh in X and $dd^c u \geq 0$ in the sense of currents ("semi-Kähler potential") on X,

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The above complex Mong-Ampère equation is understood in the weak sense of currents on X as defined by Bedford and Taylor in [BT76], [BT82].

Observe that this is a degenerate complex Monge-Ampère equation on the singular compact Kähler space V.

- The reference metric $\theta \geq 0$ is smooth and semi-positive closed $(1,1)$-form on X which vanishes along the exceptional divisors that blow down to the singular set of V but $\int_X \theta^n = \int_V \omega_V^n > 0$;

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- the volume form on the RHS μ has zeros and has poles along the exceptional divisors of π and mild singularities means that if $\mu := fdV$, dV is a smooth positive volume form on X, then $f \in L^p(X)$ for some $p > 1$.

The previous theorem follows from the next one.

Theorem

([EGZ09], [EGZ11]) Let $\theta \geq 0$ be a smooth closed semi-positive $(1,1)$-form on a compact Kähler manifold X such that $\int_X \theta^n > 0$. Let where $f \in L^p(X)$ for some $p > 1$. Then there exists a bounded θ-psh function u on X satisfying the following equation

$$(CMAE)_\varepsilon \qquad (\theta + dd^c u)^n = e^{\varepsilon u} f \, dV,$$

in the weak sense of currents on X, where dV is a smooth positive volume form on X.
Moreover u is unique when $\varepsilon > 0$ and unique up to an additive constant when $\varepsilon = 0$ and it is continuous on a Zariski open set of X (=the ample locus of the cohomology class of θ).

Comments on the proof: It is quite clear that the solution cannot be smooth du to the degeneracy of the reference metric θ and the RHS μ, so that PDE methods cannot be used.

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We will rather use tools from Pluripotential Theory. This theory was founded by E. Bedford and B.A. Taylor in [BT76], [BT82] in order to extend the classical Potential Theory to the context of several complex variables using the complex Monge-Ampère operator rather that the Laplace operator.

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This theory allows to make sense of weak solutions for degenerate complex Monge-Ampère equations using the theory of positive currents introduced by P. Lelong in the sixthies.

Later on, using the notion of capacity associated to the complex Monge-Ampère operator, S. Kolodziej [Kol98] was able to give a new proof of the fundamental a priori C^0-estimate of Yau which works for weak solutions. This allows him to extend Yau's theorem to the case when the RHS is a degenerate volume forms with L^p-density $(p > 1)$, the reference metric being Kähler.

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In [EGZ09], we extended the a priori uniform L^∞-estimates of Kolodziej to the case where both the reference metric and the volume form on the RHS are degenerate and used Yau's theorem to prove the above result by a regularization argument and a stability result for the solution.

Later on in [EGZ11] we developed a viscosity approach to solve the complex Monge-Ampère equations $(CME)_\varepsilon$ when $\varepsilon > 0$ and the density f is continuous. We discovered that suprisingly the classical Peron method can be applied to solve these equations both in the viscosity and pluripotential sense. As a consequence of this approach the solution of the equation $(CME)_\varepsilon$ $(\varepsilon > 0)$ is the upper envelope of all its subsolutions and is a bounded θ-psh function on X, continuous on the ample locus of the cohomology class of the reference metric. Letting $\varepsilon \to 0$ and using stability results we deduce that the equation $(CME)_0$ has also a weak solution with same properties. A stability argument allows to treat the case on an $L^p(X)$-density.

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