On bi-invariant linear connections on Lie groups and Symmetric Leibniz algebras

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Joint work with Mohamed Boucetta(Cadi Ayyad University, Marrakesh, Morocco).

The proofs and details of most results of this talk can be found in:

S. Benayadi and M. Boucetta,

Special bi-invariant linear connections on Lie groups and finite dimensional Poisson structures,

Journal of Differential Geometry and its Applications, Volume 36, October 2014, Pages 66-89.

All vector spaces, algebras, etc. in this talk are finite dimensional and will be over a ground field of characteristic 0.

Outline A *Poisson algebra* is a (in this talk finite dimensional) Lie algebra $(\mathfrak{g}, [\ ,\ ])$ endowed with a commutative and associative product $\circ$ such that, for any $u, v, w \in \mathfrak{g}$,

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$$[u, v \circ w] = [u, v] \circ w + v \circ [u, w]. \tag{1}$$

In this case the product given by

$$u.v = \frac{1}{2}[u, v] + u \circ v$$

is Lie admissible (because [u,v]=u.v-v.u) and satifies $[u, v.w] = [u, v].w + v.[u, w]$.

An algebra $(\mathrm{A}, .)$ is called *Poisson admissible* if $(\mathrm{A}, [\ ,\ ], \circ)$ is a Poisson algebra, where

$$[u, v] = u.v - v.u \quad \text{and} \quad u \circ v = \frac{1}{2}(u.v + v.u). \tag{2}$$

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Note that

$$u.v = \frac{1}{2}[u,v] + u \circ v. \tag{3}$$

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2. *Any Lie algebra is (trivially) Poisson admissible.*
3. *Any associative commutative algebra is Poisson admissible. In this case* $\mathfrak{g}^A$ *is abelian.*

Definitions (Leibniz Algebras)

Let $(\mathfrak{L}, .)$ be an algebra. A linear map $D : \mathfrak{L} \longrightarrow \mathfrak{L}$ is a **derivation** if

$$D(x.y) = (Dx).y + x.(Dy), \qquad \forall x, y \in \mathfrak{L}.$$

For $x \in \mathfrak{L}$, we define two linear maps:
$L_x(y) = x.y$, $\forall y \in \mathfrak{L}$ (**left multiplication** by x),
$R_x(y) = y.x$, $\forall y \in \mathfrak{L}$ (**right multiplication** by x),
$\mathrm{ad}_x := L_x - R_x$, (**the adjoint of** x).

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$(\mathfrak{L}, .)$ is a Lie algebra if $x.y = -y.x$, $\forall x, y, z \in \mathfrak{L}$ and the following equation holds

$$x.(y.z) = (x.y).z + y.(x.z), \ \forall x, y \in \mathfrak{L},$$

ie. $L_x = -R_x$ **and** L_x **is a derivation on** $\mathfrak{L}$**,** $\forall\, x \in \mathfrak{L}$.

1. The algebra $\mathfrak{L}$ is a left Leibniz algebra if the following equation holds

$$x.(y.z) = (x.y).z + y.(x.z), \qquad \forall x, y, z \in \mathfrak{L}.$$

ie. L_x **is a derivation on** $\mathfrak{L}$**, for any** $x \in \mathfrak{L}$.

2. The algebra $\mathfrak{L}$ is a right Leibniz algebra if the following equation holds

$$(x.y).z = (x.z).y + x.(y.z), \qquad \forall x, y, z \in \mathfrak{L}.$$

ie. R_x **is a derivation on** $\mathfrak{L}$**, for any** $x \in \mathfrak{L}$.

In 1965, the notion of (Left or right) Leibniz algebra was introduced by A. Bloh:

1. Bloh, A. On a generalization of the concept of a Lie algebra. Dokl. Akad. Nauk. USSR 165 (1965) 471-473.

2. Bloh, A. Cartan-Eilenberg homology theory for a generalized class of Lie algebras . Dokl. Akad.Nauk SSSR 175, 824-826 (1967),

and later, in 1993, this notion was rediscovered and developed by J-L. Loday:

Loday J-L. Une version non-commutative des algèbres de Lie: Les algèbres de Leibniz , Ens. Math., 39 (1993), 269-293.

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Proposition

$(\mathfrak{L}, .)$ is a symmetric Leibniz algebra if and only if $(\mathfrak{L}, .)$ is a left (rep. right) Leibniz algebra and $\mathfrak{L}^2 := \mathfrak{L}.\mathfrak{L} \subseteq \mathcal{A}(\mathfrak{L})$, where $\mathcal{A}(\mathfrak{L}) := \{x \in \mathfrak{L} \,/\, x.y = -y.x, \ \forall\, x, y \in \mathfrak{L}\}$.

Examples.

1. $\mathfrak{L} := \mathbb{K}X \oplus \mathbb{K}Y$ with product is defined by:

$$X.X = Y$$

   a non-Lie symmetric Leibniz algebra.

2. $\mathfrak{L} := \mathbb{K}X \oplus \mathbb{K}Y \oplus \mathbb{K}Z$ with the product is defined by:

$$X.Y = -Y.X = Z, \; X.X = Y.Y = Z$$

   a non-Lie symmetric Leibniz algebra.

Example.

Let us consider a Lie algebra $\mathfrak{g}, [\,,\,])$ a Lie algebra such that $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$ and $\varphi : \mathfrak{g} \times \mathfrak{g} \to \mathcal{V}$ a symmetric bilinear map (where $\mathcal{V}$ is a vector space) such that $\varphi([\mathfrak{g}, \mathfrak{g}], \mathfrak{g}) = \{0\}$. On the vector space $\mathfrak{L} := \mathfrak{g} \oplus \mathcal{V}$, the product:

$$(X + v).(Y + w) := [X, Y] + \varphi(X, Y),$$

$\forall\, X, Y \in \mathfrak{g}$, $v, w \in \mathcal{V}$, define a non-Lie symmetric Leibniz algebra.

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3. to show that the set $\mathcal{SL}$ of symmetric Leibniz algebras satisfies

$$\mathcal{SL} \subset \mathcal{P}^p,$$

Geometric interpretation of finite dimensional Poisson structures

Recall that a linear connection ∇ on a smooth manifold M is a $\mathbb{R}$-bilinear map

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$$

satisfying

$$\nabla_{fX} Y = f \nabla_X Y \quad \text{and} \quad \nabla_X fY = f \nabla_X Y + X(f) Y.$$

Let T^∇ and K^∇ be, respectively, the torsion and the curvature of ∇ given by

$$T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

$$K^\nabla(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Holonomy Lie algebra

For any closed curve τ at $p \in M$, let $h^\tau : T_pM \longrightarrow T_pM$ be the parallel displacement along τ. The totality of these h^τ for all closed curves forms the *holonomy group* $H(p)$. The **restricted holonomy group** $H(p)^0$ is the subgroup consisting of h^τ with τ homotopic to zero. Its Lie algebra is called **holonomy Lie algebra**.

On the other hand, consider linear endomorphisms of T_pM of the form $K^\nabla(X,Y)$, $(\nabla_Z K^\nabla)(X,Y)$, $(\nabla_W \nabla_Z K^\nabla)(X,Y)$, . . . (all covariant derivatives), where $X, Y, Z, W, \ldots$ are arbitrary tangent vectors at p. They span a subalgebra $\mathfrak{h}_p^\nabla$ of $\mathrm{End}(T_pM)$ called **infinitesimal holonomy Lie algebra**.

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The Lie subgroup $\mathrm{Exp}(\mathfrak{h}_p^\nabla)$ of $\mathrm{GL}(T_pM)$ generated by $\mathfrak{h}_p^\nabla$ is the **infinitesimal holonomy group** at p.

The main result in this theory is that:

Theorem.

$$\mathrm{Exp}(\mathfrak{h}_p^\nabla) = H(p)^0.$$

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$$K^{\overline{\nabla}}(X,Y) = K^{\nabla}(X,Y) + [S_X, S_Y] + S_{T^{\nabla}(X,Y)}. \tag{4}$$

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A connection ∇ is called **parallel** if

$$\nabla T^{\nabla} = 0 \quad \text{and} \quad \nabla K^{\nabla} = 0.$$

Now, given a manifold M with a linear connection ∇. An affine transformation is a diffeomorphism φ such that:

$$\varphi_*(\nabla_X Y) = \nabla_{\varphi_*(X)}\varphi_*(Y),$$

for any vector fields X, Y. In this case, we say that ∇ is invariant by φ
A vector field A is an infinitesimal ∇-transformation if and only if for any couple of vector fields X, Y,

$$[A, \nabla_X Y] = \nabla_{[A,X]}Y + \nabla_X[A, Y].$$

It is well known that a vector field A is infinitesimal ∇-transformation if its local flow is an affine transformation.

A linear connection ∇ on Lie group G is called left (resp. right) invariant if it is invariant by left (resp. right) multiplication. ∇ is called bi-invariant if it is left and right invariant.
If $\mathfrak{g} = T_e G$ is the Lie algebra, for any $u \in \mathfrak{g}$ we denote by u^l (resp. u^r) the left invariant (resp. the right invariant) vector field associated to u.

Proposition.

The linear connection ∇^0 *on* G *given by*

$$\nabla^0_{u^l} v^l = \frac{1}{2}[u^l, v^l],$$

for any $u, v \in \mathfrak{g}$*, is bi-invariant, torsion free, parallel, complete and its curvature and holonomy Lie algebra are given by*

$$K^{\nabla^0}(u^l, v^l)w^l = -\frac{1}{4}[[u^l, v^l], w^l], \quad u, v, w \in \mathfrak{g}. \tag{5}$$

$$\mathfrak{h}_e^{\nabla^0} = \mathrm{ad}_{[\mathfrak{g},\mathfrak{g}]}. \tag{6}$$

Lemma.

Let ∇ be a linear connection on G. Then the following assertions are equivalent:

1. *∇ is a bi-invariant linear connection.*
2. *∇ is left invariant and rigid with respect to ∇^0.*
3. *For any $X, Y \in \mathcal{X}^\ell(G)$, $\nabla_X Y \in \mathcal{X}^\ell(G)$ and the product $\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ given by*

   $$u.v = (\nabla_{u^\ell} v^\ell)(e)$$

   satisfies

   $$[u, v.w] = [u, v].w + v.[u, w]. \tag{7}$$

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$$u \circ v = u.v - \frac{1}{2}[u, v],$$

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$$[u, v \circ w] = [u, v] \circ w + v \circ [u, w].$$

Since ∇ is rigid with respect to ∇^0, (4) holds and can be written for any $u, v \in \mathfrak{g}$,

$$K^{\nabla}(u, v) = K^{\nabla^0}(u, v) + [S_u, S_v].$$

So $\circ$ is commutative and associative iff

$$T^{\nabla} = 0 \quad \text{and} \quad K^{\nabla} = K^{\nabla^0}.$$

Definition.

We call **special** *a torsion free bi-invariant linear connection on G which has the same curvature as ∇^0.*

Theorem.

Let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra. Then the following assertions hold:

1. *Let ∇ be a left invariant linear connection on G an let $\circ$ be the product on $\mathfrak{g}$ given by*

$$u \circ v = (\nabla_{u^l} v^l)(e) - \frac{1}{2}[u, v].$$

Then $(\mathfrak{g}, [\ ,\ ], \circ)$ is a Poisson algebra if and only if ∇ is special.

2. *Let $\circ$ be a product on $\mathfrak{g}$ such that $(\mathfrak{g}, [\ ,\ ], \circ)$ is a Poisson algebra. Then the linear connection on G given by*

$$\nabla_{u^l} v^l = \frac{1}{2}[u^l, v^l] + (u \circ v)^l$$

is special.

Proposition.

Any special connection ∇ *on* G *is semi-symmetric, i.e.,*

$$K.K(X,Y) := \nabla_X \nabla_Y K^{\nabla} - \nabla_Y \nabla_X K^{\nabla} - \nabla_{[X,Y]} K^{\nabla} = 0. \tag{8}$$

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Lemma.

Let ∇ *be a special connection on* G*. Then the holonomy Lie algebra of* ∇ *is given by*

$$\mathfrak{h}_e^\nabla = \mathrm{ad}_{[\mathfrak{g},\mathfrak{g}]} + \mathrm{L}_{[[\mathfrak{g},\mathfrak{g}],\mathfrak{g}]} = \mathrm{ad}_{[\mathfrak{g},\mathfrak{g}]} + \mathrm{R}_{[[\mathfrak{g},\mathfrak{g}],\mathfrak{g}]},$$

where $\mathrm{L}, \mathrm{R} : \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$ *are given by* $\mathrm{L}_u v = u.v$ *and* $\mathrm{R}_u v = v.u$ *and* $u.v = (\nabla_{u^l} v^l)(e)$.

Definition.

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Proposition.

Let ∇ a special connection on G. Then $K^{\nabla} = 0$ iff the product associated to ∇ is associative. In this case $\mathfrak{g}$ is 2-step nilpotent.

Poisson algebras: characterization and new subclasses

Let $(\mathfrak{g}, [\ ,\ ])$ be a finite-dimensional Lie algebra and $\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$, $(u, v) \mapsto u.v$ a product on $\mathfrak{g}$. For any $u \in \mathfrak{g}$, we define $\mathrm{L}_u, \mathrm{R}_u : \mathfrak{g} \longrightarrow \mathfrak{g}$ by

$$\mathrm{L}_u v = u.v \quad \text{and} \quad \mathrm{R}_u v = v.u.$$

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$$\mathrm{L}_u v = u.v \quad \text{and} \quad \mathrm{R}_u v = v.u.$$

Suppose that this product is Lie-admissible, i.e., for any $u, v \in \mathfrak{g}$,

$$u.v - v.u = [u, v].$$

Suppose also that it is bi-invariant, i.e.,

$$[u, v.w] = [u, v].w + v.[u, w].$$

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$$u.v - v.u = [u, v].$$

Suppose also that it is bi-invariant, i.e.,

$$[u, v.w] = [u, v].w + v.[u, w].$$

This is equivalent to

$$[\mathrm{L}_u, \mathrm{ad}_v] = \mathrm{L}_{[u,v]}.$$

It is obvious that the product $\circ$ on $\mathfrak{g}$ given by

$$u \circ v = u.v - \frac{1}{2}[u, v] = \frac{1}{2}(u.v + v.u) \qquad (9)$$

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Proposition.

Let $\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ a Lie-admissible product on $\mathfrak{g}$ and $\circ$ given by (9). Then $(\mathfrak{g}, [\ ,\ ], \circ)$ is a Poisson algebra if and only if, for any $u, v \in \mathfrak{g}$,

$$[\mathrm{L}_u, \mathrm{ad}_v] = \mathrm{L}_{[u,v]} \quad \textit{and} \quad K(u, v) := [\mathrm{L}_u, \mathrm{L}_v] - \mathrm{L}_{[u,v]} = -\frac{1}{4}\mathrm{ad}_{[u,v]}.$$

Definition.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra.

1. *A product . on $\mathfrak{g}$ is called* **special** *if it is Lie-admissible and, for any $u, v \in \mathfrak{g}$,*

$$[\mathrm{L}_u, \mathrm{ad}_v] = \mathrm{L}_{[u,v]} \quad \textit{and} \quad K(u,v) = -\frac{1}{4}\mathrm{ad}_{[u,v]}.$$

2. *The holonomy Lie algebra of a special product . on $\mathfrak{g}$ is the subalgebra of the Lie algebra $\mathrm{End}(\mathfrak{g})$ given by*

$$\mathfrak{h} = \mathrm{ad}_{[\mathfrak{g},\mathfrak{g}]} + \mathrm{L}_{[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]} = \mathrm{ad}_{[\mathfrak{g},\mathfrak{g}]} + \mathrm{R}_{[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]}.$$

3. *A special product . is called parallel if its curvature is parallel, i.e.,*

$$\mathrm{L}_{[u,[v,w]]} = \mathrm{ad}_{[\mathrm{L}_u v, w]} + \mathrm{ad}_{[v, \mathrm{L}_u w]}.$$

4. *A special product . is called* **strongly special** *if $\mathfrak{h} = \mathrm{ad}_{[\mathfrak{g},\mathfrak{g}]}$, i.e., $\mathrm{L}_{[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]} \subset \mathrm{ad}_{[\mathfrak{g},\mathfrak{g}]}$.*

Remark.

Note that if a product is special then its curvature vanishes if and only if it is associative. In this case, the Lie algebra $\mathfrak{g}$ *is 2-nilpotent because* $K(u,v) = -\frac{1}{4}\mathrm{ad}_{[u,v]}$.

Poisson admissible algebras: characterization and new subclasses

Proposition.

Let $(\mathrm{A},.)$ be an algebra. Then the following conditions are equivalent:

1. $(\mathrm{A},.)$ *is a Poisson admissible algebra.*
2. *For any* $u, v \in \mathrm{A}$,

$$[\mathrm{R}_u, \mathrm{R}_v] + \mathrm{L}_{[u,v]} + 3[\mathrm{L}_u, \mathrm{R}_v] = 0.$$

3. *For any* $u, v \in \mathrm{A}$,

$$[\mathrm{L}_u, \mathrm{L}_v] - \mathrm{R}_{[u,v]} + 3[\mathrm{R}_u, \mathrm{L}_v] = 0.$$

Corollary.

An associative algebra $(\mathrm{A}, .)$ *is Poisson admissible if and only if* $\mathfrak{g}^A$ *is a 2-nilpotent Lie algebra.*

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Definition.

1. *A Poisson algebra* $(A, .)$ *is called* **strongly Poisson admissible** *if it is Poisson admissible and its multiplication is strongly special on* $\mathfrak{g}^A$. *We denote by* $\mathcal{P}^s$ *the set of strongly Poisson admissible algebras.*

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2. *A Poisson algebra* $(A, .)$ *is called* **parallel Poisson admissible** *if it is Poisson admissible and its multiplication is parallel on* $\mathfrak{g}^A$*. We denote by* $\mathcal{P}^p$ *the set of parallel Poisson admissible algebras.*

Corollary.

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We have

$$\mathcal{AP} \subset \mathcal{P}^p \subset \mathcal{P}^s \subset \mathcal{P}.$$

We can state now one of our main results.

Theorem.

Let $(\mathrm{A}, .)$ *be a symmetric Leibniz algebra. Then*

$$(\mathrm{A}, .) \in \mathcal{P}^p.$$

By using the geometric interpretation of Poisson structures, we get the following interesting corollary.

Corollary.

Let $(\mathrm{A}, .)$ *be a real symmetric Leibniz algebra which is not a Lie algebra and* G *any connected Lie group associated to* $(\mathfrak{g}^{\mathrm{A}}, [\ ,\ ])$*. Then the left invariant connection on* G *given by*

$$\nabla_{u^l} v^l = (u.v)^l$$

is different from ∇^0*, strongly special and its curvature is parallel.*

An algebra $(\mathrm{A}, .)$ is called LR-algebra if, for any $u, v \in \mathrm{A}$,

$$[\mathrm{L}_u, \mathrm{L}_v] = [\mathrm{R}_u, \mathrm{R}_v] = 0.$$

It follows from Proposition 15 that a LR-algebra is Poisson admissible if and only if it is associative.

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It follows from Proposition 15 that a LR-algebra is Poisson admissible if and only if it is associative.

Proposition.

Let A *be a symmetric Leibniz and* U *an associative* LR-*algebra then* $\mathrm{A} \otimes \mathrm{U}$ *endowed with the product*

$$(u \otimes a)(v \otimes b) = (uv) \otimes (ab)$$

is a symmetric Leibniz algebra.

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It follows from Proposition 15 that a LR-algebra is Poisson admissible if and only if it is associative.

Proposition.

Let A *be a symmetric Leibniz and* U *an associative* LR-*algebra then* $\mathrm{A} \otimes \mathrm{U}$ *endowed with the product*

$$(u \otimes a)(v \otimes b) = (uv) \otimes (ab)$$

is a symmetric Leibniz algebra.

Proposition.

A left (right) Leibniz algebra is Poisson admissible if and only if it is a symmetric Leibniz algebra.

Theorem.

Let $(\mathrm{A}, .)$ *be a Poisson admissible algebra and* U *an associative* LR-*algebra. Then the product on* $\mathrm{A} \otimes \mathrm{U}$ *given by*

$$(u \otimes a) \star (v \otimes b) = \frac{1}{2}[u, v] \otimes (ab + ba) + \frac{1}{2} u.v \otimes (3ab + ba)$$

induces on $\mathrm{A} \otimes \mathrm{U}$ *a Poisson admissible algebra structure. Moreover, if* $(A, .) \in \mathcal{P}^s$ *then* $(\mathrm{A} \otimes \mathrm{U}, \star) \in \mathcal{P}^s$.

Theorem.

1. *Let* $\mathfrak{g}$ *be a perfect Lie algebra, i.e.,* $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$*. Then the product* $u.v = \frac{1}{2}[u, v]$ *is the only strongly special product on* $\mathfrak{g}$*.*
2. *Let* $\mathfrak{g}$ *be a semi-simple Lie algebra. Then the product* $u.v = \frac{1}{2}[u, v]$ *is the only special product on* $\mathfrak{g}$*. In particular, there is no non-trivial Poisson structure on* $\mathfrak{g}$*.*

Proposition.

Let $(\mathfrak{g}, [\ ,\ ])$ *be a Lie algebra and* "." *is a strongly special product on* $\mathfrak{g}$. *Then* $\mathfrak{g}^3 = [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]$ *is two sided ideal of* $(\mathfrak{g}, .)$, $(\mathfrak{g}^3, .)$ *is a symmetric Leibniz algebra and the sequence*

$$0 \longrightarrow (\mathfrak{g}^3, .) \longrightarrow (\mathfrak{g}, .) \longrightarrow (\mathfrak{g}/\mathfrak{g}^3, .) \longrightarrow 0$$

is an exact sequence of Poisson admissible algebras, $(\mathfrak{g}/\mathfrak{g}^3, .)$ *is associative and* $(\mathfrak{g}/\mathfrak{g}^3, [\ ,\ ])$ *is 2-nilpotent.*

THANK FOR YOUR ATTENTION